# Semi-Global Leader-Following Consensus of Multiple Linear Systems with Position and Rate-Limited Actuators

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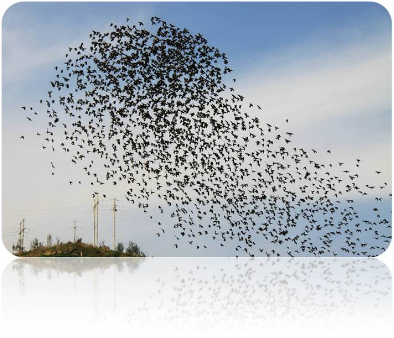
# Outline of the presentation

- Introduction
- Problem statement
- Main results
  - State feedback case
  - Output feedback case
- Concluding remarks

# Consensus of multi-agent systems

Consensus: Agents achieving an agreement on their common state by using information from the neighbors.



# Consensus of multi-agent systems

Theoretical results:

- Finite-time consensus,
- Consensus of high-order systems,
- Consensus with time delay,

$\vdots$

Practical applications:

- Autonomous underwater vehicles,
- Unmanned air vehicles,
- Wireless sensor network,

$\vdots$

# Control systems with saturating actuators - fundamental results

$$\dot{x} = Ax + Bu, \;\; x \in \mathbb{R}^n, \; ||u||_\infty \leq 1.$$

Null controllable region $\mathcal{C}$:

$$\mathcal{C} = \{x(0) \in \mathbb{R}^n : \; \exists u, \|u\|_\infty \leq 1 \text{ and } T \geq 0, \text{ s.t. } x(T) = 0\}.$$

**General characterization of $\mathcal{C}$ [Hsu, PhD Dissertation '76]**

Assume that $(A, B)$ is controllable.

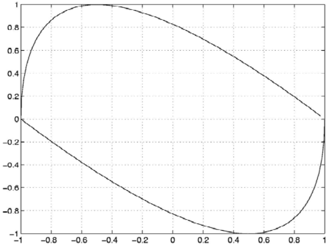
- If $A$ is semi-stable ($\lambda(A) \subset \mathbf{C}^- \cup \mathbf{C}^0$), then, $\mathcal{C} = \mathbb{R}^n$.
- If $A$ is anti-stable ($\lambda(A) \subset \mathbf{C}^+$), then, $\mathcal{C}$ is a bounded convex open set.
- If $A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$, $B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$, $A_1 \in \mathbb{R}^{n_1 \times n_1}$, $A_2 \in \mathbb{R}^{n_2 \times n_2}$, $\lambda(A_1) \subset \mathbf{C}^+$, $\lambda(A_2) \subset \mathbf{C}^- \cup \mathbf{C}^0$, then, $\mathcal{C} = \mathcal{C}_1 \times \mathbb{R}^{n_2}$, where $\mathcal{C}_1$ is the controllable region of $\dot{x}_1 = A_1 x_1 + B_1 \sigma(u)$.

# Control systems with saturating actuators - fundamental results

## Characterization of $\mathcal{C}_1$

T. Hu and Z. Lin, *Control Systems with Actuator Saturation: Analysis and Design*, Birkhauser, 2001.

$$A = \begin{bmatrix} 0 & -0.5 \\ 1 & 1.5 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \Rightarrow \partial\mathcal{C} = \left\{ \pm \left( -2e^{-At} + I \right) A^{-1} B : \ t \in [0, \infty] \right\}$$



# Control systems with saturating actuators - fundamental results

## Global and semi-global stabilization

- Global stabilization requires $(A, B)$ to be asymptotically null controllable with bounded controls (ANCBC), *i.e.*, $(A, B)$ is stabilizable and $\lambda(A) \subset \mathbf{C}^- \cup \mathbf{C}^0$ [Sussmann, Sontag & Yang, *CDC* '90]
- in general, nonlinear feedback laws are needed [Fuller, IJC '76; Sussmann & Yang, *CDC* '91]
- nonlinear, but smooth, stabilizers [Sussmann, Sontag & Yang; Teel; Megretski; Lin; $\cdots$, '90s]
- Semi-global stabilization can be achieved with linear feedback [Lin, *Low Gain Feedback*, Springer, '98]

# Consensus of multi-agent systems subject to input saturation [Meng, Zhao & Lin, *SCL* '13, Zhao & Lin, *CCC* '13]

Consider a group of $N$ networked follower agents subject to actuator saturation:

$$\dot{x}_i = Ax_i + B\sigma(u_i),\ i = 1, 2, \cdots, N,$$

where $x_i \in \mathrm{R}^n$ is the state, $u_i \in \mathrm{R}^m$ is the input, $(A, B)$ is stabilizable, and $\sigma(u_i) = [\sigma(u_{i1}), \sigma(u_{i2}), \cdots, \sigma(u_{im})]^{\mathrm{T}}$, with $\sigma(u_{ij}) = \mathrm{sign}(u_{ij}) \min\{|u_{ij}|, \Delta\}$.

The trajectory of the leader agent is governed by

$$\dot{x}_0 = Ax_0.$$

# Consensus of multi-agent systems subject to input saturation [Meng, Zhao & Lin, *SCL* '13, Zhao& Lin, *CCC* '13]

## Global leader-following consensus problem

Design a local feedback law for each follower agent such that

$$\lim_{t\to\infty}(x_i(t) - x_0(t)) = 0, \ i = 1, 2, \cdots, N.$$

## Semi-global leader following consensus problem

For any given bounded set $\mathcal{X} \subset \mathrm{R}^n$, design a local feedback law for each follower agent such that, for all $x_i(0) \in \mathcal{X}$, $i = 0, 1, 2, \cdots, N$,

$$\lim_{t\to\infty}(x_i(t) - x_0(t)) = 0, \ i = 1, 2, \cdots, N.$$

## Related problems

Consensus problems for multi-agent systems subject to simultaneous actuator position and rate saturation.

# Semi-global stabilization of linear systems with position and rate-limited actuators [Lin, *SCL* '97]

Consider a linear system with both position and rate-limited actuators,

$$\begin{cases} \dot{x} = Ax + B\sigma_{\mathrm{p}}(v), \\ \dot{v} = \sigma_{\mathrm{r}}(-v + u), \\ y = Cx, \end{cases}$$

is semi-globally asymptotically stabilizable by linear state feedback if the open loop system is asymptotically null controllable with bounded controls, that is $(A, B)$ is stabilizable in the usual linear systems theory sense and all eigenvalues of $A$ are on the closed left-half plane. If, in addition, $(A, C)$ is detectable, then the system is semi-globally asymptotically stabilizable by linear output feedback.

# Problem statement

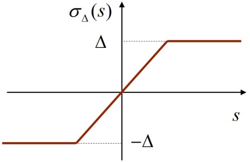
Consider a group of $N$ networked follower agents with position and rate-limited actuators,

$$\begin{cases} \dot{x}_i = Ax_i + B\sigma_{\rm p}(v_i), \\ \dot{v}_i = \sigma_{\rm r}(-v_i + u_i), \quad i = 1, 2, \cdots, N, \end{cases}$$

where saturation functions $\sigma_{\rm p}(s)$ and $\sigma_{\rm r}(s)$ are respectively defined as $\sigma_{\rm p}(s) = \mathrm{sign}(s)\min\{\Delta_{\rm p}, |s|\}$ and $\sigma_{\rm r}(s) = \mathrm{sign}(s)\min\{\Delta_{\rm r}, |s|\}$, for some $\Delta_{\rm p} > 0$ and $\Delta_{\rm r} > 0$.

The trajectory of the leader agent is governed by

$$\dot{x}_0 = Ax_0.$$



# Problem statement

**Semi-global consensus by linear state feedback**

For any *a priori* given bounded sets $\mathcal{X}_0 \subset \mathbb{R}^n$ and $\mathcal{V}_0 \subset \mathbb{R}^m$, construct a linear state feedback control law $u_i$ for each follower agent, which only uses local information, such that all these feedback control laws together achieve semi-global leader-following consensus, that is, for all $x_i(0) \in \mathcal{X}_0$, $i = 0, 1, \cdots, N$, and $v_i(0) \in \mathcal{V}_0$, $i = 1, 2, \cdots, N$,

$$\lim_{t\to\infty}(x_i(t) - x_0(t)) = 0, i = 1, 2, \cdots, N.$$

# Problem statement

## Semi-global consensus by linear output feedback

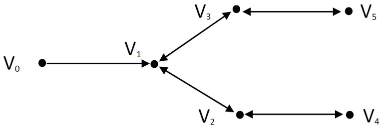
For any *a priori* given bounded sets $\mathcal{X}_0 \subset \mathbb{R}^n$ and $\mathcal{V}_0 \subset \mathbb{R}^m$, construct a linear observer based output feedback control law $u_i$ for each follower agent, which only uses local information, such that all these feedback control laws together achieve semi-global leader-following consensus, that is, for all $x_i(0), \hat{x}_i(0) \in \mathcal{X}_0$, $i = 0, 1, \cdots, N$, and $v_i(0) \in \mathcal{V}_0$, $i = 1, 2, \cdots, N$,

$$\lim_{t\to\infty}(x_i(t) - x_0(t)) = 0, i = 1, 2, \cdots, N,$$

where $\hat{x}$ is the state of the observer.

# Graph theory

$$
\begin{aligned}
\mathcal{G}_N = \{\mathcal{V}, \mathcal{E}\} &: \text{Graph} \\
\mathcal{V} = \{\nu_1, \nu_2, \cdots, \nu_N\} &: \text{Nodes} \\
\mathcal{E} \in \mathcal{V} \times \mathcal{V} &: \text{Edges} \\
(v_{i1}, v_{i2}), (v_{i2}, v_{i3}), \cdots &: \text{Path} \\
A_N = [a_{ij}] &: \text{Adjacency matrix} \\
L_N = [l_{ij}] &: \text{Laplacian matrix} \\
\text{diag}\{a_{10}, a_{20}, \cdots, a_{N0}\} &: \text{Connection between followers and leader}
\end{aligned}
$$




An undirected graph

# Preliminaries

**Assumption 1**

The undirected graph $\mathcal{G}_N$ is connected and $a_{i0} > 0$ for at least one $i, i = 1, 2, \cdots, N$.

Denote $M = L_N + \mathrm{diag}\{a_{10}, a_{20}, \cdots, a_{N0}\}$.

**Lemma 1 [Hu & Hong, *Physica A* '07]**

Let Assumption 1 hold. Then, $M$ is symmetric and positive definite.

For the positive definite matrix $M$, we order its eigenvalues as $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$.

# Preliminaries

## Assumption 2

The pair $(A, B)$ is stabilizable and all eigenvalues of $A$ are on the closed left-half plane.

## Lemma 2[Lin, *Low Gain Feedback*, London'98]

Let Assumption 2 hold. Then, for each $\varepsilon \in (0, 1]$, there exists a unique matrix $P(\varepsilon) > 0$ that solves the algebraic Riccati equation (ARE)

$$A^{\mathrm{T}}P + PA - PBB^{\mathrm{T}}P + \varepsilon I = 0.$$

Moreover, such a $P(\varepsilon)$ satisfies

1. $\lim_{\varepsilon \to 0} P(\varepsilon) = 0$;
2. There exists a constant $\alpha > 0$, independent of $\varepsilon$, such that
$$\left\| P^{\frac{1}{2}}(\varepsilon) A P^{-\frac{1}{2}}(\varepsilon) \right\| \le \alpha, \ \varepsilon \in (0, 1].$$

# Preliminaries

## Corollary 1

Let Assumptions 1 and 2 hold. Then, there exists a unique positive definite solution $P(\varepsilon)$ to the following algebraic Riccati equation

$$A^{\mathsf{T}}P + PA - \gamma PBB^{\mathsf{T}}P + \varepsilon I = 0, \quad \varepsilon \in (0, 1]$$

where $\gamma$ is any positive constant such that $\gamma \le \lambda_1$.

## Lemma 3

Let Assumptions 1 and 2 hold. Then, for any vector $x \in \mathrm{R}^{Nn}$, the following inequality always holds,

$$x^{\mathsf{T}}(M^2 \otimes PBB^{\mathsf{T}}P)x \ge \gamma x^{\mathsf{T}}(M \otimes PBB^{\mathsf{T}}P)x,$$

where $\gamma \le \lambda_1$.

# Semi-global consensus by linear state feedback

## Control laws:

$$u_i = -\frac{1}{\varepsilon^2} B^{\mathrm{T}} P(\varepsilon) \left( \sum_{j=1}^{N} a_{ij}(x_i - x_j) + a_{i0}(x_i - x_0) \right) - \left( \frac{1}{\varepsilon^2} - 1 \right) v_i,$$

$$i = 1, 2, \cdots, N.$$

## Theorem 1

Let Assumptions 1 and 2 hold. Then, under the linear state feedback control laws, the group of follower agents and the leader agent achieve semi-global leader-following consensus. That is, for any given bounded sets $\mathcal{X}_0 \subset \mathbb{R}^n$ and $\mathcal{V}_0 \subset \mathbb{R}^m$, there is an $\varepsilon^* > 0$ such that, for any $\varepsilon \in (0, \varepsilon^*]$,

$$\lim_{t \to \infty} (x_i(t) - x_0(t)) = 0, \quad i = 1, 2, \cdots, N,$$

for all $x_i(0) \in \mathcal{X}_0, i = 0, 1, \cdots, N$, and $v_i(0) \in \mathcal{V}_0, i = 1, 2, \cdots, N$.

# Proof of Theorem 1

Let the error states be $\bar{x}_i = x_i - x_0, i = 1, 2, \cdots, N$. Denote $\bar{x} = [\bar{x}_1^{\mathrm{T}}, \bar{x}_2^{\mathrm{T}}, \cdots, \bar{x}_N^{\mathrm{T}}]^{\mathrm{T}}$, $v = [v_1^{\mathrm{T}}, v_2^{\mathrm{T}}, \cdots, v_N^{\mathrm{T}}]^{\mathrm{T}}$ and $u = [u_1^{\mathrm{T}}, u_2^{\mathrm{T}}, \cdots, u_N^{\mathrm{T}}]^{\mathrm{T}}$. Then we have

$$\begin{cases} \dot{\bar{x}} = (I_N \otimes A)\bar{x} + (I_N \otimes B)\sigma_{\mathrm{p}}(v), \\ \dot{v} = \sigma_{\mathrm{r}}(-v + u). \end{cases}$$

Notice that the state feedback control laws can be written as

$$u = -\frac{1}{\varepsilon^2}\left(v + (M \otimes B^{\mathrm{T}} P)\bar{x}\right) + v.$$

Construct a Lyapunov function,

$$V(\bar{x}, v) = \bar{x}^{\mathrm{T}}(M \otimes P)\bar{x} + \left(v + (M \otimes B^{\mathrm{T}} P)\bar{x}\right)^{\mathrm{T}}\left(v + (M \otimes B^{\mathrm{T}} P)\bar{x}\right),$$

which is positive definite since $M$ and $P$ are both positive definite.

# Proof of Theorem 1

Let $c > 0$ be a constant scalar such that

$$c \geq \sup_{\varepsilon\in(0,1], x_i\in\mathcal{X}_0, i=0,1,\cdots,N, v_i\in\mathcal{V}_0, i=1,2,\cdots,N} V(\bar{x}, v).$$

Such a $c$ exists since $\mathcal{X}_0$ and $\mathcal{V}_0$ are both bounded and $\lim_{\varepsilon\to 0} P(\varepsilon) = 0$.

Let $L_V(c) := \left\{(\bar{x}, v) \in \mathbb{R}^{N(n+m)} : V(\bar{x}, v) \leq c\right\}$. Let $\varepsilon_1^* \in (0, 1]$ be such that for all $\varepsilon \in (0, \varepsilon_1^*]$, $(\bar{x}, v) \in L_V(c)$ implies that

$$\begin{aligned}
\|(M \otimes B^{\mathrm{T}} P(\varepsilon))\bar{x}\| &\leq \frac{\Delta}{3}, \\
\|(M \otimes B^{\mathrm{T}} P(\varepsilon)A)\bar{x}\| &\leq \frac{\Delta}{3Nm}, \\
\|(M \otimes B^{\mathrm{T}} P(\varepsilon)B)\sigma_{\mathrm{p}}(v)\| &\leq \frac{\Delta}{3Nm},
\end{aligned}$$

where $\Delta = \min\{\Delta_{\mathrm{p}}, \Delta_{\mathrm{r}}\}$.

# Proof of Theorem 1

The derivative of $V$ along the trajectories of the closed-loop system inside the level set $L_V(c)$ can be evaluated as follows,

$$\dot{V} \leq -\varepsilon \bar{x}^{\mathrm{T}}(M \otimes I_n)\bar{x} + 2\sum_{k=1}^{Nm} \left( -\mu^k \left( \sigma_{\mathrm{p}}\left(v^k\right) - \mu^k \right) + \left(v^k - \mu^k\right) \right.$$
$$\left. \times \left( \sigma_{\mathrm{r}}\left( -\frac{1}{\varepsilon^2}\left(v^k - \mu^k\right)\right) + F^k \bar{x} + K^k \sigma_{\mathrm{p}}(v) \right) \right),$$

where $v^k$ is the $k$th element of $v = [v_1^{\mathrm{T}}, v_2^{\mathrm{T}}, \cdots, v_N^{\mathrm{T}}]^{\mathrm{T}}$, $\mu^k$ is the $k$th element of $\mu = -(M \otimes B^{\mathrm{T}}P)\bar{x}$, $F^k$ is the $k$th column of matrix $F = (M \otimes B^{\mathrm{T}}PA)$ and $K^k$ is the $k$th column of matrix $(M \otimes B^{\mathrm{T}}PB)$.

Discuss the derivative of $V$ under the following three cases:

- $|v^k - \mu^k| > \varepsilon^2 \Delta$ for all $k = 1, 2, \cdots, Nm$,
- $|v^k - \mu^k| < \varepsilon^2 \Delta$ for at least one $k$ but not all $k$, $k = 1, 2, \cdots, Nm$,
- $|v^k - \mu^k| \leq \varepsilon^2 \Delta$ for all $k = 1, 2, \cdots, Nm$.

We arrive at the conclusion that, there exists an $\varepsilon^* \in (0, \varepsilon_1^*]$ such that, for all $\varepsilon \in (0, \varepsilon^*]$,

$$\dot{V} < 0, \forall(\bar{x}, v) \in L_v(c) \setminus \{0\}.$$

This implies that the closed-loop system is asymptotically stable at $(\bar{x}, v) = (0, 0)$ with $L_V(c)$ included in the domain of attraction, and hence,

$$\lim_{t\to\infty} (x_i(t) - x_0(t)) = 0, \quad i = 1, 2, \cdots, N,$$

hold for all $x_i(0) \in \mathcal{X}_0, i = 0, 1, \cdots, N$ and $v_i(0) \in \mathcal{V}_0, i = 1, 2, \cdots, N$.

# Semi-global consensus by linear output feedback

**Assumption 3**

The pair $(A, C)$ is detectable.

Construct a state observer for each agent as follows,

$$\dot{\hat{x}}_i = A\hat{x}_i + L(y_i - C\hat{x}_i), \ i = 0, 1, \cdots, N,$$

where $\hat{x}_i \in \mathbb{R}^n$ and $L$ is any matrix such that $A - LC$ is Hurwitz.

**Control laws:**

$$u_i = -\frac{1}{\varepsilon^2} B^{\mathrm{T}} P(\varepsilon) \left( \sum_{j=1}^{N} a_{ij}(\hat{x}_i - \hat{x}_j) + a_{i0}(\hat{x}_i - \hat{x}_0) \right) - \left( \frac{1}{\varepsilon^2} - 1 \right) v_i, \quad i = 1, 2, \cdots, N.$$

# Semi-global consensus by linear output feedback

## Theorem 2

Let Assumptions 1, 2 and 3 hold. Then, under the linear output feedback control laws, the group of follower agents and the leader agent achieve semi-global leader-following consensus. That is, for any given bounded sets $\mathcal{X}_0 \subset \mathbb{R}^n$ and $\mathcal{V}_0 \subset \mathbb{R}^m$, there is an $\varepsilon^* > 0$ such that, for any given $\varepsilon \in (0, \varepsilon^*]$,

$$\lim_{t \to \infty} (x_i(t) - x_0(t)) = 0, \quad i = 1, 2, \cdots, N,$$

for all $x_i(0), \hat{x}_i(0) \in \mathcal{X}_0, i = 0, 1, \cdots, N$, and $v_i(0) \in \mathcal{V}_0, i = 1, 2, \cdots, N$.

# Proof of Theorem 2

Let $e_i = x_i - \hat{x}_i, i = 0, 1, \cdots, N$. We have

$$\begin{cases} \dot{e}_0 = (A - LC)e_0, \\ \dot{e}_i = (A - LC)e_i + B\sigma_{\mathrm{p}}(v_i), \ i = 1, 2, \cdots, N. \end{cases}$$

Let $\bar{e}_i = e_i - e_0, i = 1, 2, \cdots, N$, then we have

$$\dot{\bar{e}}_i = (A - LC)\bar{e}_i + B\sigma_{\mathrm{p}}(v_i), \ i = 1, 2, \cdots, N.$$

Denote $\bar{x}_i = x_i - x_0, \bar{y}_i = y_i - y_0, i = 1, 2, \cdots, N$. Then we have

$$\begin{cases} \dot{\bar{x}}_i = A\bar{x}_i + B\sigma_{\mathrm{p}}(v_i), \\ \dot{v}_i = \sigma_{\mathrm{r}}(-v_i + u_i), \\ \dot{\bar{e}}_i = (A - LC)\bar{e}_i + B\sigma_{\mathrm{p}}(v_i), \\ u_i = -\frac{1}{\varepsilon^2}B^{\mathrm{T}}P\Big(\sum_{j=1}^{N} a_{ij}((\bar{x}_i - \bar{x}_j) - (\bar{e}_i - \bar{e}_j)) + a_{i0}(\bar{x}_i - \bar{e}_i)\Big) - \big(\frac{1}{\varepsilon^2} - 1\big)v_i, \\ \qquad i = 1, 2, \cdots, N. \end{cases}$$

# Proof of Theorem 2

Let $\bar{x} = [\bar{x}_1^\mathsf{T}, \bar{x}_2^\mathsf{T}, \cdots, \bar{x}_N^\mathsf{T}]^\mathsf{T}$, $v = [v_1^\mathsf{T}, v_2^\mathsf{T}, \cdots, v_N^\mathsf{T}]^\mathsf{T}$, $u = [u_1^\mathsf{T}, u_2^\mathsf{T}, \cdots, u_N^\mathsf{T}]^\mathsf{T}$, and $\bar{e} = [\bar{e}_1^\mathsf{T}, \bar{e}_2^\mathsf{T}, \cdots, \bar{e}_N^\mathsf{T}]^\mathsf{T}$. Then we have

$$\begin{cases} \dot{\bar{x}} = (I_N \otimes A)\bar{x} + (I_N \otimes B)\sigma_{\mathrm{p}}(v), \\ \dot{v} = \sigma_{\mathrm{r}}(-v + u), \\ \dot{\bar{e}} = (I_N \otimes (A - LC))\bar{e} + (I_N \otimes B)\sigma_{\mathrm{p}}(v), \\ u = -\frac{1}{\varepsilon^2}\left(M \otimes B^\mathsf{T} P\right)(\bar{x} - \bar{e}) - \left(\frac{1}{\varepsilon^2} - 1\right) v. \end{cases}$$

Construct a Lyapunov function,

$$\begin{aligned} V(\bar{x}, v, \bar{e}) = {} & \bar{x}^\mathsf{T}(M \otimes P)\bar{x} + \lambda_{\max}^{1/2}(P)\bar{e}^\mathsf{T}(M \otimes P_0)\bar{e} + \left(v + (M \otimes B^\mathsf{T} P)\,\bar{x}\right. \\ & \left. - (M \otimes B^\mathsf{T} P)\,\bar{e}\right)^\mathsf{T}\left(v + (M \otimes B^\mathsf{T} P)\,\bar{x} - (M \otimes B^\mathsf{T} P)\,\bar{e}\right), \end{aligned}$$

where $P_0 > 0$ is the unique positive definite solution to the following Lyapunov equation,

$$(A - LC)^\mathsf{T} P_0 + P_0(A - LC) = -I.$$

# Proof of Theorem 2

Let $c > 0$ be a constant scalar such that

$$c \geq \sup_{\varepsilon \in (0,1], x_i, \hat{x}_i \in \mathcal{X}_0, i=0,1,\cdots,N, v_i \in \mathcal{V}_0, i=1,2,\cdots,N} V(\bar{x}, v, \bar{e}).$$

Such a $c$ exists since $\mathcal{X}_0$ and $\mathcal{V}_0$ are both bounded and $\lim_{\varepsilon \to 0} P(\varepsilon) = 0$.

Let $L_V(c) := \left\{(\bar{x}, v, \bar{e}) \in \mathbb{R}^{N(2n+m)} : V(\bar{x}, v, \bar{e}) \leq c\right\}$. Let $\varepsilon_1^* \in (0, 1]$ be such that, for all $\varepsilon \in (0, \varepsilon_1^*]$, $(\bar{x}, v, \bar{e}) \in L_V(c)$ implies that,

$$\|(M \otimes B^{\mathrm{T}} P)\, \bar{x}\| \leq \frac{\Delta}{8}, \quad \|(M \otimes B^{\mathrm{T}} P)\, \bar{e}\| \leq \frac{\Delta}{8}, \quad \|(M \otimes B^{\mathrm{T}} PA)\, \bar{x}\| \leq \frac{\Delta}{8Nm},$$

$$\left\|\lambda_{\max}^{1/2}(P)\, (M \otimes B^{\mathrm{T}} P_0)\, \bar{e}\right\| \leq \frac{\Delta}{8}, \quad \|(M \otimes B^{\mathrm{T}} P(A - LC))\, \bar{e}\| \leq \frac{\Delta}{8Nm},$$

where $\Delta = \min\{\Delta_{\mathrm{p}}, \Delta_{\mathrm{r}}\}$.

# Proof of Theorem 2

The derivative of $V$ inside the level set $L_V(c)$ can be evaluated as follows,

$$\dot{V} \leq -\varepsilon \bar{x}^{\mathrm{T}}(M \otimes I_n)\bar{x} - \lambda_{\max}^{1/2}(P)\bar{e}^{\mathrm{T}}(M \otimes I_n)\bar{e} + 2\sum_{k=1}^{Nm}\left(-\lambda_{\max}^{1/2}(P)\eta^k \sigma_{\mathrm{p}}\left(v^k\right)\right.$$

$$\left. -\frac{1}{4}\left(\phi^k\right)^2\right) + 2\sum_{k=1}^{Nm}\left(-\frac{1}{4}\left(\phi^k\right)^2 - \phi^k\left(\sigma_{\mathrm{p}}\left(v^k\right) - \phi^k\right)\right.$$

$$\left. + \left(v^k - \phi^k + \omega^k\right)\left(\sigma_{\mathrm{r}}\left(-\frac{1}{\varepsilon^2}\left(v^k - \phi^k + \omega^k\right)\right) + G^k\bar{x} + H^k\bar{e}\right)\right),$$

where $v^k$ is the $k$th element of $v$, $\eta^k$ is the $k$th element of $\eta = -\left(M \otimes B^{\mathrm{T}}P_0\right)\bar{e}$, $\phi^k$ is the $k$th element of $\phi = -\left(M \otimes B^{\mathrm{T}}P\right)\bar{x}$, $\omega^k$ is the $k$th element of $\omega = -\left(M \otimes B^{\mathrm{T}}P\right)\bar{e}$, $G^k$ is the $k$th column of matrix $G = M \otimes B^{\mathrm{T}}PA$, and $H^k$ is the $k$th column of matrix $H = M \otimes B^{\mathrm{T}}P(A - LC)$.

# Proof of Theorem 2

Discuss the derivative of $V$ under the following three cases:

- $\left|v^k - \phi^k + \omega^k\right| > \varepsilon^2 \Delta$ for all $k = 1, 2, \cdots, Nm$,
- $|\left|v^k - \phi^k + \omega^k\right| > \varepsilon^2 \Delta$ for at least one $k$ but not all $k, k = 1, 2, \cdots, Nm$,
- $\left|v^k - \phi^k + \omega^k\right| \leq \varepsilon^2 \Delta$ for all $k = 1, 2, \cdots, Nm$.

We can conclude that, there exists an $\varepsilon \in (0, \varepsilon_1^*]$ such that, for all $\varepsilon \in (0, \varepsilon^*]$,

$$\dot{V} < 0, \forall (\bar{x}, v) \in L_v(c) \setminus \{0\}.$$

This implies that the closed-loop system is asymptotically stable at $(\bar{x}, v) = (0, 0)$ with $L_V(c)$ included in the domain of attraction, and hence,

$$\lim_{t \to \infty} (x_i(t) - x_0(t)) = 0, \quad i = 1, 2, \cdots, N,$$

hold for all $x_i(0) \in \mathcal{X}_0, i = 0, 1, \cdots, N$, and $v_i(0) \in \mathcal{V}_0, i = 1, 2, \cdots, N$.

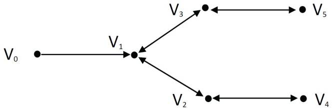
# An example

Consider a group of 5 agents and a leader with

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix},$$

and $\Delta_{\mathrm{p}} = 5, \Delta_{\mathrm{r}} = 0.5$.

The communication topology among agents is as shown below:



# An example

For the given graph, we have

$$M = \begin{bmatrix} 3 & -1 & -1 & 0 & 0 \\ -1 & 2 & 0 & -1 & 0 \\ -1 & 0 & 2 & 0 & -1 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix}.$$

The minimum eigenvalue of $M$ is $\lambda_{\min}(M) = 0.1392$.

We choose $\gamma = 0.01 < \lambda_{\min}(M)$.

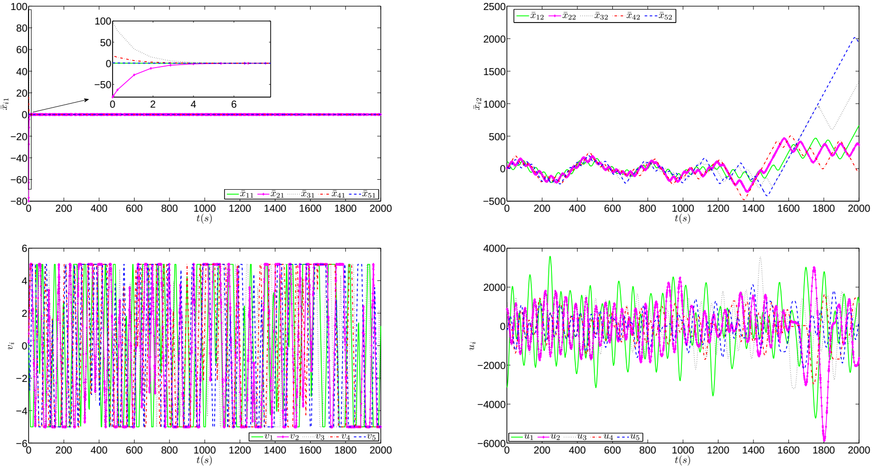
# Simulation results - state feedback

Choose the initial values of the agents randomly as

$$
\begin{aligned}
&\begin{bmatrix} x_1(0) & x_2(0) & x_3(0) & x_4(0) & x_5(0) & x_0(0) \end{bmatrix} \\
&= \begin{bmatrix} -10 & 0.1 & -80 & 98 & 18 & 1 \\ 10 & 108 & 10 & -0.1 & -0.5 & 20 \end{bmatrix}, \\
&\begin{bmatrix} v_1(0) & v_2(0) & v_3(0) & v_4(0) & v_5(0) \end{bmatrix} \\
&= \begin{bmatrix} 0.1 & 0.2 & 0.3 & 0.4 & 0.5 \end{bmatrix}.
\end{aligned}
$$

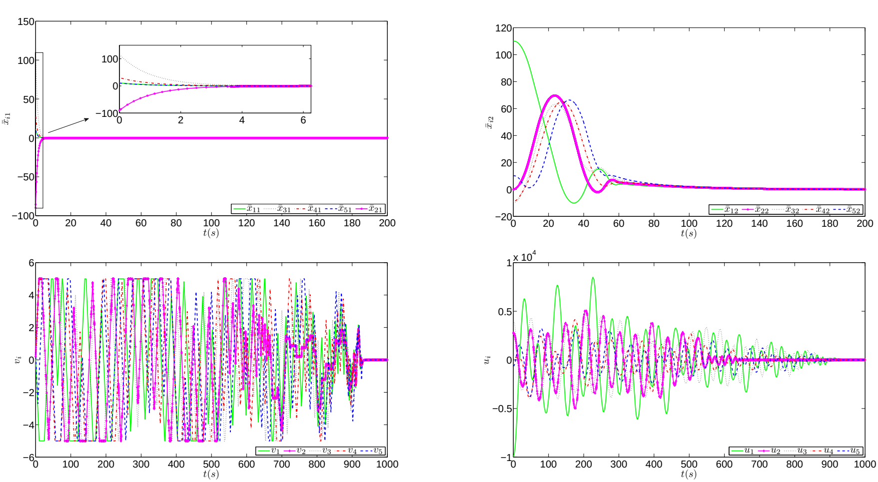
We will simulate the closed-loop system for two different values of $\varepsilon$, $\varepsilon = 1$ and $\varepsilon = 0.1$.

# Simulation results - state feedback




Evolutions of the agents under state feedback control laws when $\varepsilon = 1$.

# Simulation results - state feedback




Evolutions of the agents under state feedback control laws when $\varepsilon = 0.1$.

# Simulation results - output feedback

Choose the initial values of the agents randomly as

$$\begin{bmatrix} x_1(0) & x_2(0) & x_3(0) & x_4(0) & x_5(0) & x_0(0) \end{bmatrix} = \begin{bmatrix} -10 & 0.1 & -80 & 98 & 18 & 1 \\ 10 & 108 & 10 & -0.1 & -0.5 & 20 \end{bmatrix},$$

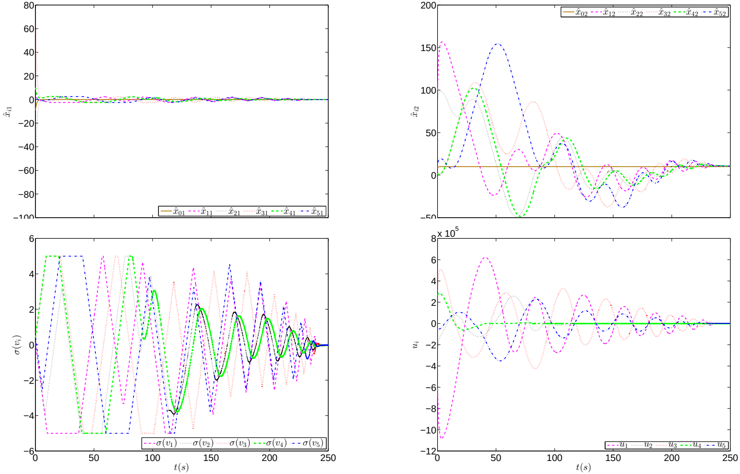
$$\begin{bmatrix} v_1(0) & v_2(0) & v_3(0) & v_4(0) & v_5(0) \end{bmatrix} = \begin{bmatrix} 0.1 & 0.2 & 0.3 & 0.4 & 0.5 \end{bmatrix}.$$

Choose the initial values of observer of the agents randomly as

$$\begin{bmatrix} \hat{x}_0(0) & \hat{x}_1(0) & \hat{x}_2(0) & \hat{x}_3(0) & \hat{x}_4(0) & \hat{x}_5(0) \end{bmatrix} = \begin{bmatrix} -9 & 10 & -90 & 70 & 10 & -1 \\ 9 & 100 & 90 & 1 & 2 & 13 \end{bmatrix}.$$

We will simulate the closed-loop system $\varepsilon = 0.01$.

# Simulation results - output feedback



Evolutions of the agents under output feedback control laws with $\varepsilon = 0.01$.

# Conclusions

We studied the semi-global leader-following consensus problem for a group of linear systems in the presence of both actuator position and rate saturation.

We constructed both a family of linear state feedback control laws and a family of linear output feedback control laws for each follower agent by using low gain feedback design strategy, which only uses the information of agent and its neighbors.

Semi-global leader-following consensus can be achieved by using the proposed control laws when the communication topology among follower agents is a connected undirected graph and the leader is a neighbor of at least one follower.

Great challenges remain when the agents are open loop exponentially unstable.

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