Distributed Constrained Optimal Control of Multi-agent Systems with Application

presented by

Professor Lihua Xie (谢立华)
School of EEE, Nanyang Technological University, Singapore
Email: elhxie@ntu.edu.sg

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Outline

- **Consensus based Distributed Optimization and Control**

  Problem Formulation and Motivations

  Introduction to subgradient methods and consensus

  Distributed optimization

    by decomposition of decision vector

    by combining consensus

  Distributed optimal control with continuous-time dynamics

- **Distributed MPC with application to formation flight**
- **UAV and UGV Demos**
- **Conclusion**

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Distributed Optimization: Model & Problem

- A network of m agents with node set $N=\{1,\ldots,m\}$

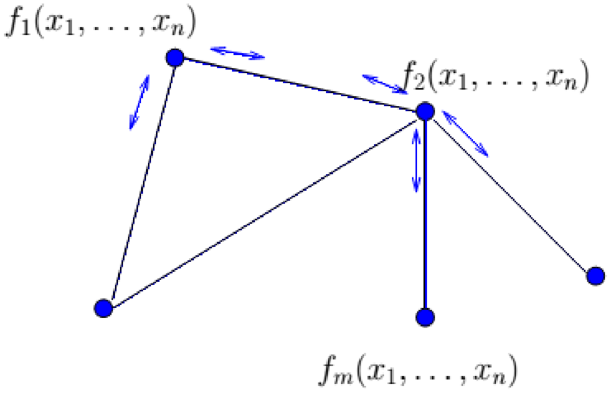
  -- cooperatively solve

$$\min F(x)=\sum_{i=1}^{m} f_i(x)$$

$$\text{s.t. } g(x)=(g_1(x),\ldots,g_s(x))'\leq 0,$$

$$h(x)=(h_{s+1}(x),\ldots,h_t(x))'=0,$$

$$x\in X=\bigcap_{i=1}^{m} X_i$$



  -- local cost function $f_i: R^n \to R$ is convex and known to agent i

  -- $g_j: R^n \to R$ is convex; h_j is affine; local constraint X_i is convex and closed; X is nonempty

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Motivation of distributed optimization and control

- Global information is difficult to access in large-scale networks, and centralized algorithms are not robust against unexpected changes in topology

- "Big Data" problems consist of millions or billions of training samples, and the data is often collected and stored in a distributed manner

- Distributed algorithms only rely on local observation and information, hence can react efficiently to time-varying topology and deal with large-scale problems

- Applications: task assignment, congestion control, distributed estimation, machine learning

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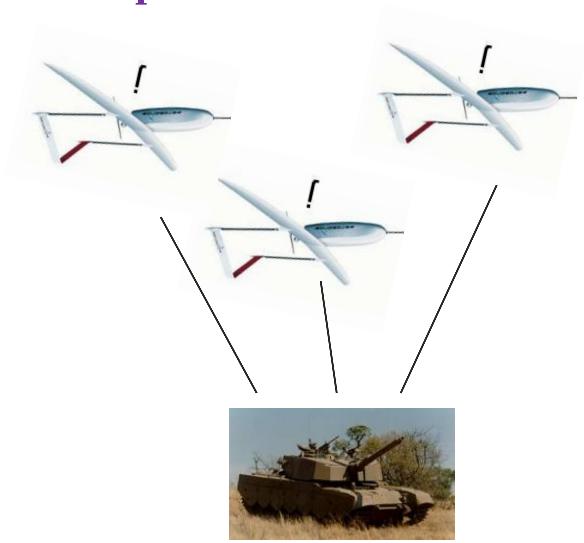
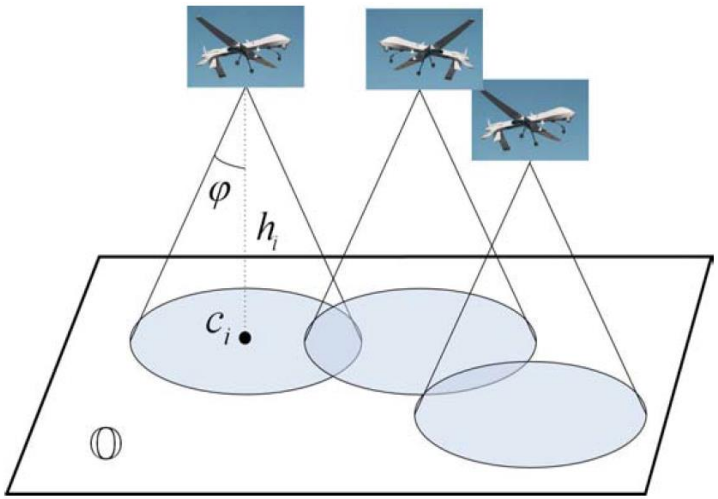
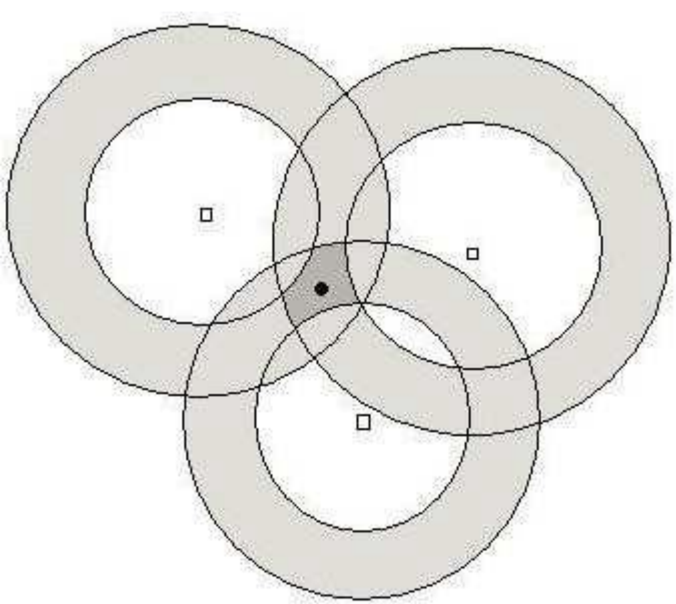
Examples: Distributed Localization and Coverage Control

Cooperative localization



Coverage Control





$$\min_P \mathcal{H}(P) = \min_P \int_Q F(q, P, t)\phi(q, t)dq$$

$$\dot{p}_i = g_i(p, w) + h_i(p)u_i, \;\; p_i(0) = p_{i0}, \;\; \|u_i\| < u_{max}$$

$$\lambda_i(p_i, q, t) \leq \Lambda_i, \;\; \sum_i \lambda_i(p_i, q, t) \leq \Lambda$$

$$\varrho_i \circ F(q, P, t) \leq \Upsilon_i, \;\; \rho_i \circ \phi(q, t) \leq \Psi_i$$

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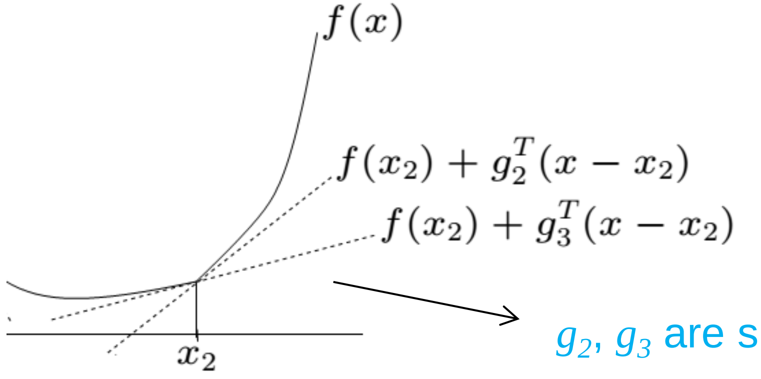
Basics of convex functions

- Convex set C: $tx+(1-t)y \in C,\ \forall x, y \in C, 0<t<1$
- For a closed and convex set $C \subseteq R^n$, $P_C(x) = \arg\min_{c\in C} \| x-c \|_2$ is the projection of x onto C
- $f: R^n \to R$ is convex if $f(tx+(1-t)y) \le tf(x)+(1-t)f(y),\ 0<t<1,\ \forall x, y \in R^n$
  A convex function is differentiable almost everywhere.
- The subdifferential of f at x is defined by

$$\partial f(x) = \{s \mid s'(y-x) \le f(y) - f(x),\ \forall y \in R^n\}$$

with $s \in S$ called as the subgradient . $\partial f(x)$ is nonempty, compact and convex everywhere; as a set-valued map, it is also upper semi-continuous.



g_2, g_3 are subgradients at x_2

Primal and dual problem

- Primal problem

$$\min F(x), \text{ s.t. } g(x) \le 0,\ h(x) = 0,\ x \in X$$

with optimal value f^* and optimal point x^*

- Dual problem

$$\max q(\lambda), \text{ s.t. } \lambda = (\lambda_1, \dots, \lambda_s, \lambda_{s+1}, \dots, \lambda_t)' \in K = R_+^s \times R^{t-s}$$

$$L(x,\lambda) = F(x) + \lambda'[g(x)', h(x)']',\ q(\lambda) = \min_{x \in X} L(x,\lambda)$$

with optimal value q^* and optimal point λ^*

- $f^* = q^*$ and (x^*, λ^*) is the optimal pair if and only if $L(x^*,\lambda) \le L(x^*,\lambda^*) \le L(x,\lambda^*)$ (Saddle-point Condition)

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Subgradient Methods

- Projected subgradient method for primal problem with set-constraint

$$x(k+1) = P_X[x(k) - \alpha(k)d(k)],\ d(k) \in \partial F(x(k))$$

- Projected subgradient method for dual problem

$$x(k+1) = \arg\min_{x \in X} L(x, \lambda(k)),$$
$$\lambda(k+1) = P_K[\lambda(k) + \alpha(k)\underline{g(x(k+1))}]$$

↓ subgradient of $q(\lambda(k))$

⇨ No guarantee for convergence in many applications, esp when the minimization step fails

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Variants of dual subgradient method

- Projected subgradient method to approximate the saddle point (Nedić & Ozdaglar, 2009)

$$x(k+1) = P_X[x(k) - \alpha(k) d_x(k)],\ d_x(k) \in \partial_x L(x(k), \lambda(k));$$

$$\lambda(k+1) = P_M[\lambda(k) + \alpha(k) d_\lambda(k)],\ d_\lambda(k) \in \partial_\lambda L(x(k), \lambda(k)).$$

$\lambda^* \in M$; M convex and compact.

- Augmented Lagrangians and the method of multipliers

$$\min f(x),\ \text{s.t. } h(x) = 0 \Leftrightarrow \min f(x) + \rho \| h(x) \|^2 /2,\ \text{s.t. } h(x) = 0$$

Augmented Lagrangian: $L_\rho(x, \lambda) = f(x) + \lambda' h(x) + \boxed{\rho \| h(x) \|^2 /2}$

$$x(k+1) = \arg\min_{x \in X} L_\rho(x, \lambda(k)),$$

$$\lambda(k+1) = \lambda(k) + \rho h(x(k+1))$$

Fixed step size

Quadratic penalty added to guarantee the minimization step, and improve convergence properties

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- Alternating Direction Method of Multipliers (ADMM)

$$\min f_1(x) + f_2(z), \text{ s.t. } h_1(x) + h_2(z) = 0$$

$$L_\rho(x, z, \lambda) = f_1(x) + f_2(z) + \lambda'(h_1(x) + h_2(z)) + \rho \| h_1(x) + h_2(z) \|^2 / 2$$

$$x(k+1) = \arg\min_x L_\rho(x, z(k), \lambda(k)),$$

$$z(k+1) = \arg\min_z L_\rho(x(k+1), z, \lambda(k));$$

$$\lambda(k+1) = \lambda(k) + \rho(h_1(x(k+1)) + h_1(z(k+1)))$$

⇨ Two-step minimization

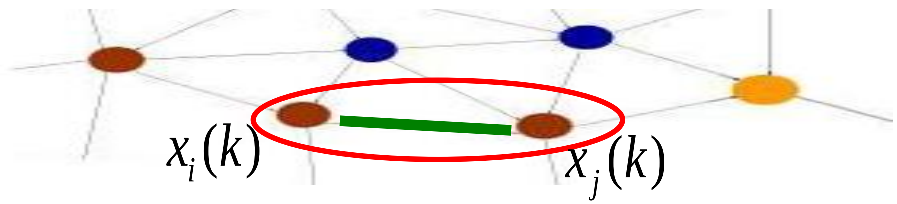
Introduction to Consensus

- Distributed consensus: Networked agents to reach a common state by exchanging information with neighbors

$$x_i(k+1) = x_i(k) + u_i(k), \quad u_i(k) = \sum_{j \in N_i} a_{ij}(x_i(k) - x_j(k))$$

$$x_i(k) - x_j(k) \to 0, \forall i, j$$

$$G = \{V, E, A\}$$



- Reformulated as an optimization problem

$$\min \frac{1}{2} \sum_{i,j} a_{ij} \| x_i - x_j \|^2$$

- Applications: coordination, rendezvous, formation control, swarming

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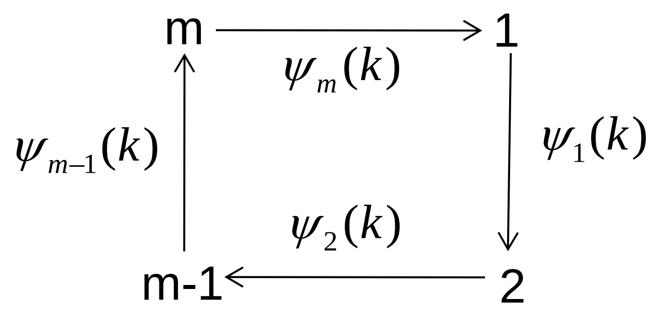
Distributed optimization by decomposition --Incremental subgradient methods

- Deterministic algorithm by cycling (Nedic & Bertsekas, 2001)

$$
\begin{aligned}
&x(k+1) = \psi_m(k) \\
&\psi_1(k) = P_X[x(k) - \alpha(k) d_1(k)],\ d_1(k) \in \partial f_1(x(k)); \\
&\psi_2(k) = P_X[\psi_1(k) - \alpha(k) d_2(k)],\ d_2(k) \in \partial f_2(\psi_1(k)); \\
&\dots. \\
&\psi_m(k) = P_X[\psi_{m-1}(k) - \alpha(k) d_m(k)],\ d_m(k) \in \partial f_m(\psi_{m-1}(k));
\end{aligned}
$$

Each node updates the decision vector based on local subgradient, and passes it to the next node



A. Nedic and D. P. Bertsekas, "Incremental subgradient methods for nondifferentiable optimization," *SIAM Journal on Optimization, vol. 12, pp. 109-138, 2001..*

- Randomized algorithm

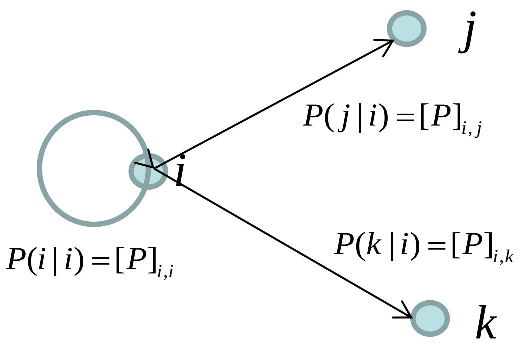
$$x(k+1) = P_X[x(k) - \alpha(k) d_{w(k)}(k)],\ d_{w(k)}(k) \in \partial f_{w(k)}(x(k))$$

-- Choosing w(k) from {1,…,m} by probability 1/m (Nedic & Bertsekas, 2001)

-- Choosing w(k) from a Markov process (Johansson, et. al., 2009)

$$[P]_{i,j} = \begin{cases} \min(1/d_i, 1/d_j),\ (i,j) \in \varepsilon, i \neq j; \\ 1 - \sum_{j \neq i} [P]_{i,j},\ i = j \\ 0,\ \text{otherwise.} \end{cases}$$

d_i is the number of edges of node i



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B. Johansson, M. Rabi, and M. Johansson, "A randomized incremental subgradient method for distributed optimization in networked systems," *SIAM Journal on Optimization, vol. 20, pp. 1157-1170, 2009.*

Distributed ADMM

- Unconstrained optimization (f_i strongly convex and coercive)

$$\min \sum_{i=1}^{m} f_i(x) \Leftrightarrow \min \sum_{i=1}^{m} f_i(x_i), \text{ s.t. } x_1 = x_2, \ldots, x_{m-1} = x_m, x_m = x_1$$

- Decomposition of primal variables and dual variables

$$L_\rho(x_1, \ldots, x_m, \lambda_1, \ldots, \lambda_m) = \sum_{i=1}^{m} [\, f_i(x_i) + \lambda_i (x_i - x_{i+1}) + \rho \| x_i - x_{i+1} \|^2 / 2 \,] \quad (x_{m+1} = x_1)$$

--In a cyclic implementation, node i keeps the primal variable x_i and dual primal λ_i and updates the pair with the information from neighbors:

$$x_i^{k+1} = \arg\min_{x_i} \; f_i(x_i) + (\lambda_i^k - \lambda_{i-1}^k) + \rho(\| x_i - x_{i+1}^k \|^2 + \| x_i - x_{i-1}^{k+1} \|^2),$$

$$\lambda_i^{k+1} = \lambda_i^k + \rho(x_i^{k+1} - x_{i+1}^k)$$

--A similar scheme also applies in a general connected network.

-- x_i^k, λ^k respectively converge to the optimal pair

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E. Wei and A. E. Ozdaglar, "Distributed Alternating Direction Method of Multipliers," in *CDC, 2012, pp. 5445-5450.*

Distributed Optimization by using consensus method

- Reformulation of the problem

$$\min F(x) = \sum_{i=1}^{m} f_i(x)$$
$$\text{s.t. } g(x) \le 0,\ h(x) = 0,$$
$$x \in X = \bigcap_{i=1}^{m} X_i$$

$\Leftrightarrow$

$$\min \sum_{i=1}^{m} f_i(x_i)$$
$$\text{s.t. } x_1 = \ldots = x_n,$$
$$g(x_i) \le 0, h(x_i) = 0,\ x_i \in X_i$$

- An asymptotic solution of the problem

$$\lim_{t\to\infty} \| x_i(t) - x_j(t) \| = 0,\ \forall i, j;$$
$$\lim_{t\to\infty} | x_i(t) |_{X^*} = 0,\ X^* = \{x \mid \arg\min_{x\in X, g(x)\le 0, h(x)=0} F(x)\}.$$

- Distributed projected subgradient method with set constraint but without inequality constraint (Nedic, et. al., 2010; Johansson, et. al., 2008)

Consensus $v_i(k)=\sum_{j=1}^{m} a_{ij}(k)x_j(k),$

⬇

Update $x_i(k+1)=P_{X_i}[v_i(k)-\alpha(k)d_i(k)],\ d_i(k)\in\partial f_i(v_i(k))$

Alternatively

$$v_i(k)=x_i(k)-\alpha(k)d_i(k),\ d_i(k)\in\partial f_i(v_i(k))$$

$$x_i(k+1)=P_{X_i}[\sum_{j=1}^{m} a_{ij}(k)v_j(k)],$$

-- $X_i=X$; bounded subgradients

-- Non-degenerate weights: $a_{ii}(k)>\eta>0;\ a_{ij}(k)>\eta$ if $a_{ij}(k)>0.$

-- Doubly stochasticity: $\sum_{j=1}^{m} a_{ij}(k)=\sum_{i=1}^{m} a_{ij}(k)=1$

-- Periodically jointly strongly connected topology

-- Non-summable but square-summable stepsize:

$\sum_k \alpha(k)=\infty,\ \sum_k \alpha^2(k)<\infty$ (constant step size results in static error)

$\Rightarrow$ convergence to the bounded optimal set

Nedic, A. Ozdaglar, and P. A. Parrilo, "Constrained consensus and optimization in multi-agent networks," TAC*2010.*
B. Johansson, et.al., "Subgradient methods and consensus algorithms for solving convex optimization problems," *2008. CDC*

- Distributed primal-dual subgradient method to approximate the saddle point (Zhu & Martínez, 2012)

  -- Step 1: Averaging

$$v_i^x(k) = \sum_{j=1}^{m} a_{ij}(k) x_j(k),\ v_i^\lambda(k) = \sum_{j=1}^{m} a_{ij}(k)\lambda_j(k)$$

  -- Step 2: Projection of subgradients

$$x(k+1) = P_{X_i}[v_i^x(k) - \alpha(k) d_i^x(k)],\ d_i^x(k) \in \partial_x L(v_i^x(k), v_i^\lambda(k));$$

$$\lambda(k+1) = P_{M_i}[v_i^\lambda(k) + \alpha(k) d_i^\lambda(k)],\ d_i^\lambda(k) \in \partial_\lambda L(v_i^x(k), v_i^\lambda(k)).$$

$\lambda^* \in M_i;\ M_i$ convex and compact.

$\Rightarrow$ convergence to the optimal pair (x^*, λ^*)

under similar assumptions

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M. Zhu and S. Martínez, "On distributed convex optimization under inequality and equality constraints," *Automatic Control, IEEE Transactions on, vol. 57, pp. 151-164, 2012.*

Distributed Optimization in Other Scenarios

- Quantization effect (Rabbat & Nowak, 2005)
- Asynchronous communication (Nedic, 2011)
- Random communication graph (Lobel & Ozdaglar , 2011)
- Stochastic subgradient errors (Ram et. al., 2010)
- Approximate projection (Lou et. al., 2012)

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Motivation

- In many practical problems, consensus is required to meet some (constrained) optimal criterion, e.g. a group of UAVs seeking for a rendezvous location in some presubscribed area while minimizing the total travelling distance
- The algorithm of distributed optimization cannot be applied directly: in numeric calculation, the projection of each agent's state onto the constraint set is feasible, but a moving agent is unable to move into the constraint set immediately
- Special cases

  -- Minimizing the sum of local cost functions with a nonempty intersection of optimal sets (G. Shi, et.al, 2012)

  -- Computing the intersection of convex sets (G. Shi, et.al, 2013)

G. Shi, A. Proutiere, and K. H. Johansson, "Distributed Optimization: Convergence Conditions from a Dynamical System Perspective," *arXiv preprint arXiv:1210.6685, 2012.*

G. Shi, K. H. Johansson, and Y. Hong, "Reaching an optimal consensus: dynamical systems that compute intersections of convex sets," *Automatic Control, IEEE Transactions on, vol. 58, pp. 610-622, 2013.*

Distributed optimal control with a common set constraint

- Problem formulation $\min \sum_{i=1}^{m} f_i(x_i),\ \text{s.t. } x_1 = \ldots = x_m \in X$
- Distributed control input

$$\dot{x}_i(t) = u_i(t) \in \sum_j a_{ij}(t)(x_j - x_i) + P_{X_i}(x_i) - x_i - \alpha(t)\partial f_i(x_i)$$

- The optimal consensus based control can be achieved asymptotically under the following assumptions:
  - -- X closed and convex
  - -- f_i convex and coercive
  - -- balanced, uniformly jointly connected graph
  - -- $0 < a_* \leq a_{ij}(t) \leq a^*$ if $a_{ij}(t) > 0$
  - -- $\lim_{t\to\infty} \alpha(t) = 0,\ \int_0^\infty \alpha(t)dt = \infty$
- Difference with the projected subgradient method: parallel projection term; no bound for subgradient term;

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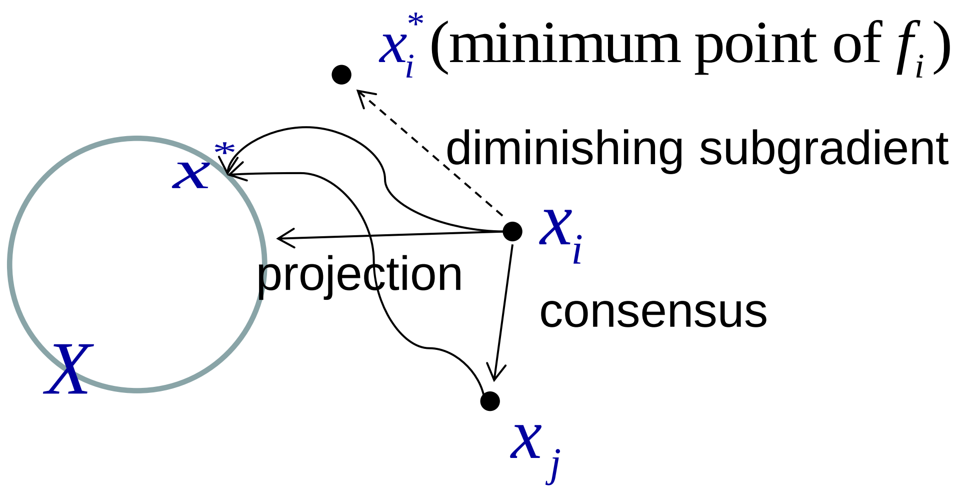
Z. Qiu, S. Liu, and L. Xie, “Distributed Constrained Optimal Control of Multi-agent Systems,” submitted to Automatica

An insight of the problem

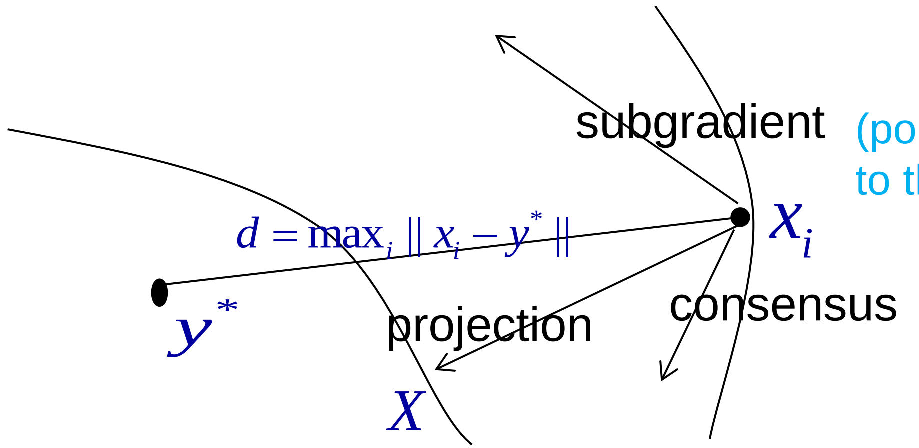


--Three different forces: consensus, projection, diminishing but persistent subgradient

-- Once the solution is bounded, the first two forces are dominant and lead to a consensus in the constraint set; the last one asymptotically drives the agents to the optimal set within the constraint set

Sketch of Proof

- 1. Boundedness of the solution



(points into the circle due to the coercity)

- 2. Constraint set convergence

$$\text{bounded solution} \Rightarrow |\partial f_i(x_i(t))| \le s^*, \forall i$$

$$p(t) = \max_i |x_i(t)|_X^2$$

$$D^+ p(t) \le -2p(t) + 2\alpha(t)s^*\sqrt{p(t)},\ \alpha(t) \to 0 \Rightarrow p(t) \to 0$$

- 3. Consensus analysis

$$\dot{x}_i(t)=\sum_j a_{ij}(t)(x_j-x_i)+P_{X_i}(x_i)-x_i-\alpha(t)s_i(t),\ s_i(t)\in\partial f_i(x_i(\mathrm{t}))$$

Diminishing disturbance

- 4. Optimal set convergence

$$l(t)=\sum_i |x_i(t)|^2_{X^*},\ X^* \text{ is the optimal set of } F=\sum_{i=1}^{m} f_i \text{ over } X$$

$$\bar{x}=\sum_i x_i/m;\ \forall\varepsilon>0,\exists\, t_\varepsilon \text{ such that}$$

$$\dot{l}(t)\le 2\alpha(t)[F(P_{X^*}(\bar{x}))-F(P_X(\bar{x}))+s^*\varepsilon],t\ge t_\varepsilon$$

$$\Rightarrow\int_0^\infty \alpha(t)dt=\infty \text{ ensures } \liminf_{t\to\infty} l(t)\to 0;$$

$$\text{Whennever } l(t) \text{ increases, } F(P_X(\bar{x}))\le F(P_{X^*}(\bar{x}))+s^*\varepsilon$$

$$\Rightarrow\lim_{t\to\infty} F(P_X(\bar{x}))=F(P_{X^*}(\bar{x}))$$

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Fixed Unbalanced Graph Case

- Algorithm based on matrix scaling technique

$$\dot{r}_i(t) = \sum_{j=1}^{n} a_{i,j} r_j(t), r_i(0) = 1,$$

$$\dot{x}_i(t) = \sum_{j=1}^{n} a_{i,j} r_j(t) x_j(t) - \alpha(t) \nabla f_i(x_i) + P_{\mathcal{X}}(x_i) - x_i, i = 1, \ldots, n,$$

- $r(t) = [r_1(t), \cdots r_n(t)]'$ converges to the left eigenvector of Laplacian matrix corresponding to eigenvalue 0
- Under the same assumption, the optimization problem can be solved under unbalanced digraph

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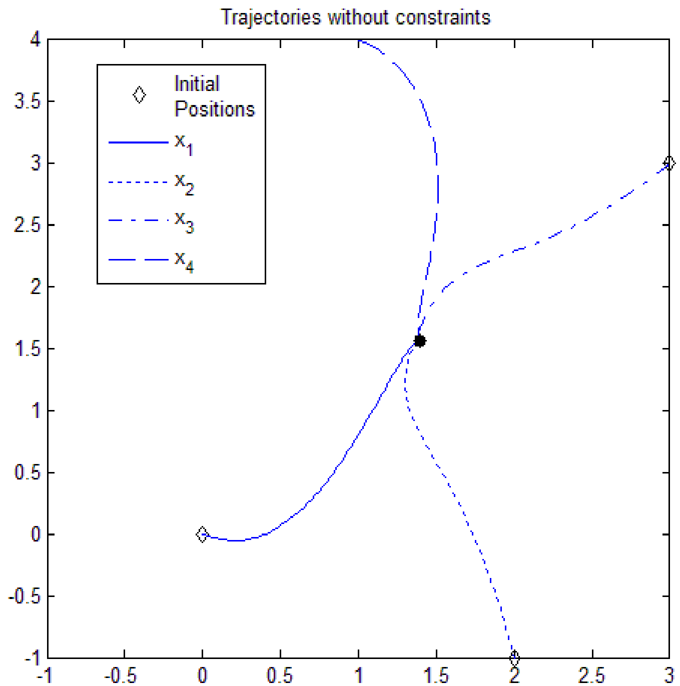
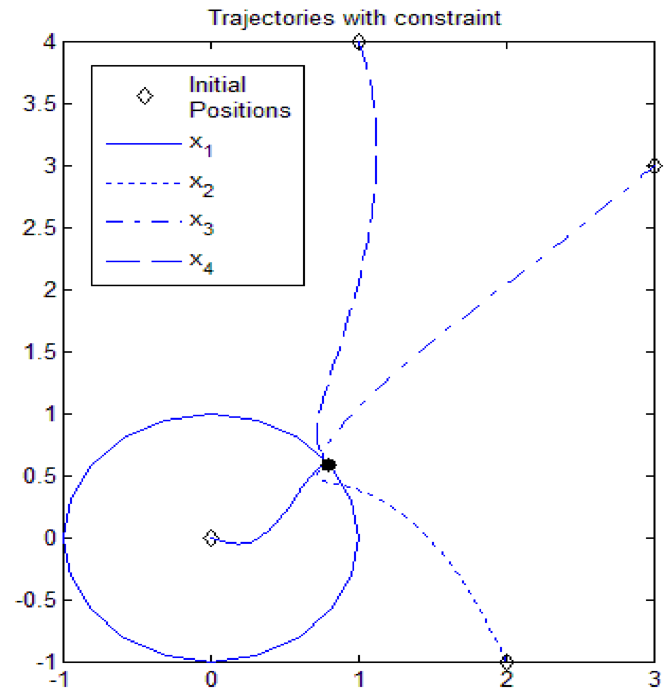
Simulation Results





Topology: $1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow 4$ $\quad f_i(x_i) = |x_i - x_i(0)|$ $\quad X = \overline{B(0,1)}$

$$x_1(0) = [0,0],\ x_2(0) = [2,-1],\ x_3(0) = [3,3],\ x_4(0) = [1,4],$$

$\alpha(t) = 1/[t^{0.8}\ln(t+1)]$ neither integrable nor square summable

Convergence Speed

- If $F(x)$ is strongly convex with parameter $c>0$, i.e.

$$F(tx+(1-t)y) \le tF(x)+(1-t)F(y)-\frac{c}{2}t(1-t)\|x-y\|^2, t\in[0,1]$$

then the optimal point x^* is unique and $\dot{l}(t) \le -\frac{c}{m}\alpha(t)l(t)+M\alpha^2(t)$

(1) If $\limsup_{t\to\infty}(-\dot{\alpha}/\alpha^2) < c/m$, then $\|x_i(t)-x^*\|^2 \sim O(\alpha(t))$,
e.g. $\alpha(t)=1/t^\beta, \beta\in(0,1)$

(2) If $\limsup_{t\to\infty}(-\dot{\alpha}/\alpha^2) = b < \infty$ and $b > c/m$, then

$$\|x_i(t)-x^*\|^2 \sim O(\alpha(t)^{\frac{c}{mb}})$$

e.g. $\alpha(t)=1/t, c/m<1 \Rightarrow \|x_i(t)-x^*\|^2 \sim O(1/t^{c/m})$

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Further discussion

- Different constraint X_i for different agent is to be considered
- The inclusion of an integral term may accelerate the convergence speed

  -- An example of unconstrained optimization (B. Gharesifard & J. Cortés, 2014)

$$\begin{cases} \dot{x}_i(t) \in \sum_j a_{ij}[(x_j - x_i) + (z_j - z_i)] - \partial f_i(x_i), \\ \dot{z}_i(t) = \sum_j a_{ij}(x_j - x_i) \end{cases}$$

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B. Gharesifard and J. Cortés, "Distributed continuous-time convex optimization on weight-balanced digraphs," *Automatic Control, IEEE Transactions on, vol. 59, pp. 781-786, 2013.*

A discrete-time implementation

- Assumptions:

  --A1: X closed and convex

  --A2: f_i convex and coercive
- Algorithm

$$\begin{cases} s_i(k) = x_i(k) + h(P_{\mathcal{X}}(x_i(k)) - x_i(k) - \alpha(k)\nabla f_i(x_i(k))), \\ x_i(k+1) = s_i(k) + h\sum_{j\in\mathcal{V}} a_{i,j}(s_j(k) - s_i(k)),\ k = 1, 2, \ldots, \end{cases}$$

- With a fixed graph and Assumptions A1, A2, the above algorithm asymptotically solves the distributed optimization problem with a sufficiently small step size h if and only if

  -- The graph is balanced and contains a spanning tree.

  -- $\lim_{k\to\infty} \alpha(k) = 0,\ \sum_{k=1}^{\infty} \alpha(k) = \infty$

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Discussion on the Necessity Part

- The graph has to contain a spanning tree. Otherwise the nodes can be divided into two groups with no communication between them.
- The graph has to be balanced. Otherwise the convergence value becomes the solution of

$$\min F(x) = \sum_{i=1}^{m} r_i f_i(x), \text{ s.t. } x \in X$$

where $r(t) = [r_1(t), \cdots r_n(t)]'$ is a left eigenvector of Laplacian matrix corresponding to eigenvalue 0

- If $\lim_{t\to\infty} \alpha(t) \neq 0$, the set constraint cannot be fulfilled.
- If $\int_0^\infty \alpha(t)dt \neq \infty$, the states prematurely stop outside the optimum set

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Convergence Speed

- The following hold if $F(x)$ is strongly convex with parameter $c>0$:

$$(1)\ \ \| x_i(k) - x^* \|^2 \sim O(\alpha(k)) \text{ when } \limsup_{k\to\infty} \frac{\alpha(k)-\alpha(k+1)}{h\alpha^2(k)} < c/m$$

$$(2)\ \ \| x_i(k) - x^* \|^2 \sim O(\alpha(k)^{\frac{hc}{mb}}) \text{ when } \limsup_{k\to\infty} \frac{\alpha(k)-\alpha(k+1)}{h\alpha^2(k)} < b < \infty \text{ and } b > c/m$$

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Event-Triggered Based Algorithm

- Event-based algorithm (even-triggered comm.)

$$s_i(k) = x_i(k) + h(P_{\mathcal{X}}(x_i(k)) - x_i(k) - \alpha(k)\nabla f_i(x_i(k)))$$
$$x_i(k+1) = s_i(k) + h\sum_{j\in\mathcal{V}} a_{ij}(t)(s_j(k_t) - s_i(k))$$

- Trigger function for k_{t+1}: $s_i(k) - s_i(k_t) \geq M\alpha(k)$
- Under the same assumption and F is strongly convex, the optimization problem is solved with static error cM ($c > 0$ is a constant)

$$\lim_{k\to\infty} |x_i(k) - x_j(k)|^2 = 0$$
$$\limsup_{k\to\infty} |x_i(k)|^2_{\mathcal{X}^*} \leq cM$$

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Random Optimization

- Consider noises exist in calculating subgradient

$$s_i(k) = x_i(k) + h(P_{\mathcal{X}}(x_i(k)) - x_i(k) - \alpha(k)\nabla f_i(x_i(k) + \varepsilon_i(k)))$$
$$x_i(k+1) = s_i(k) + h\sum_{j\in\mathcal{V}} a_{ij}(t)(s_j - s_i)$$

- **Assumption:**

- Noises $\varepsilon_i(k)$ are zero mean white noises with bounded variances
- The objective functions are globally Lipschitz

- The problem is solved in mean square sense

$$\lim_{k\to\infty} E\,|\,x_i(k) - x_j(k)\,|^2 = 0$$
$$\lim_{k\to\infty} E\,|\,x_i(k)\,|^2_{\mathcal{X}^*} = 0$$

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Outline

- **Consensus based Distributed Optimization**
  - Problem Formulation and Motivations
  - Introduction to subgradient methods and consensus
  - Distributed optimization
    - by decomposition of decision vector
    - by combining consensus
  - Distributed optimal control with continuous-time dynamics
- **Distributed MPC with application to formation flight**
- **UAV and UGV Demos**
- **Conclusion**

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MPC based Formation Flight Control

MPC offers a number of unique advantages:

- Deal with the constrained MIMO dynamics of the UAV system by directly using its mathematical model in the control loop design
- Consider the formation flight kinematics and dynamics of the UAV as an entire system, which results in an integrated formation flight framework;
- Give a local path planning function by combining future reference and the environment information such as obstacles

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Existing Works

- The update sequence of the UAV
  - Richards, A. and J. How, 2004
  - Kuwata, Y. and J.P. How, 2010
  - Chung, H., 2006
  - Keviczky, T., F. Borrelli, K. Fregene, D. Godbole, and G. Balas, 2008
- Collision avoidance scheme in the MPC framework
  - Xu, B., D.J. Stilwell, and A.J. Kurdila, 2010
  - Kuwata, Y. and J.P. How, 2010
  - Bellingham, J., M. Tillerson, M. Alighanbari, and J. How, 2002
- The safety flight maneuver envelope is not guaranteed

Scenario & Objectives

- Only the leader is given the maneuver commands
- The follower aircraft should:
  - Maintain the formation by
    - Following the changes in the leader states
    - Suppressing disturbances
  - Avoid collision with each other and with external obstacles

Objectives:

Develop formation flight control system based on model predictive control (MPC) to enable collision free formation flight. During the formation flight, all the agents should be able to keep the specified formation in the presence of disturbances and uncertainty while avoiding collision with each other and with the obstacles..

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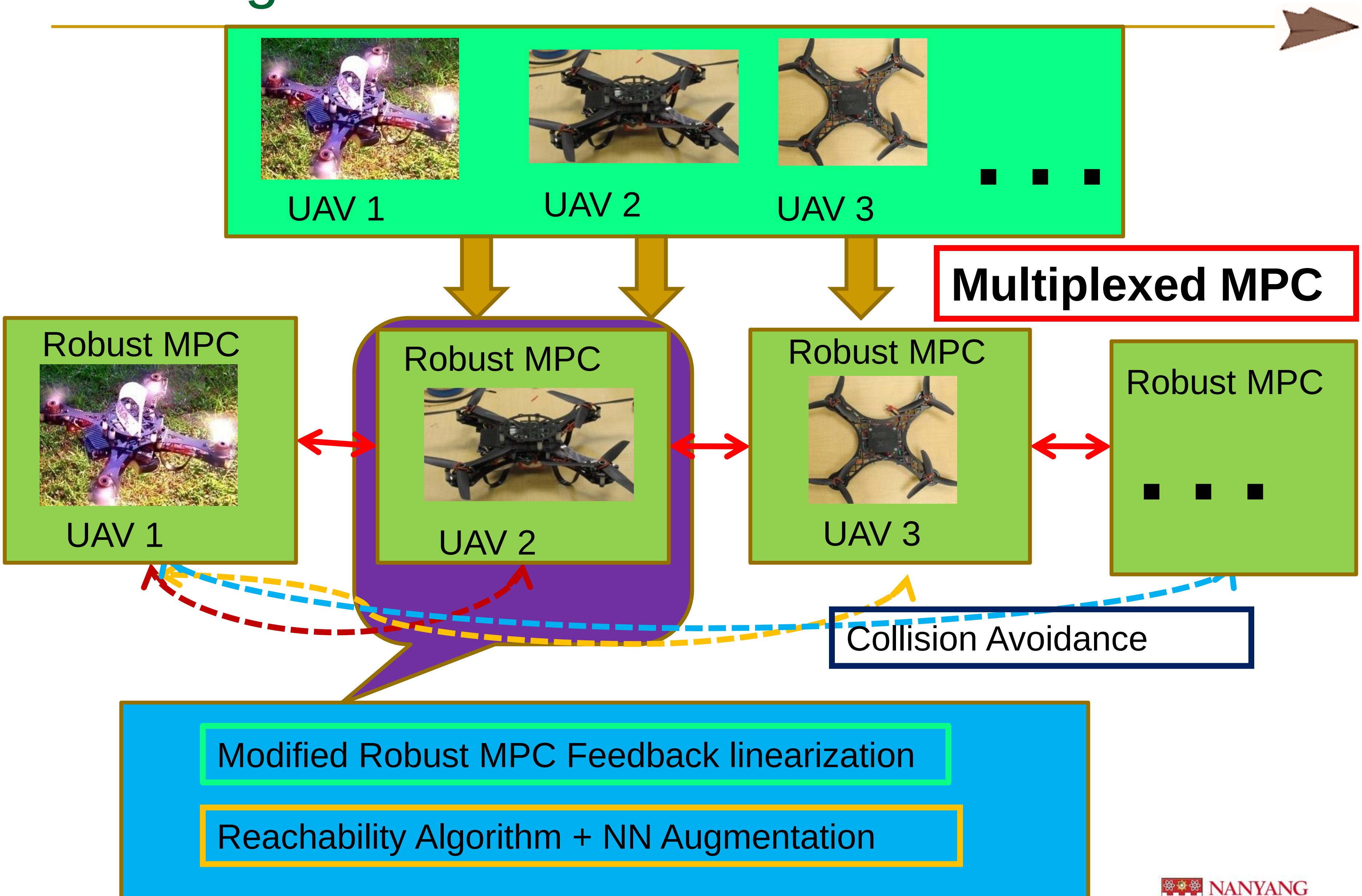
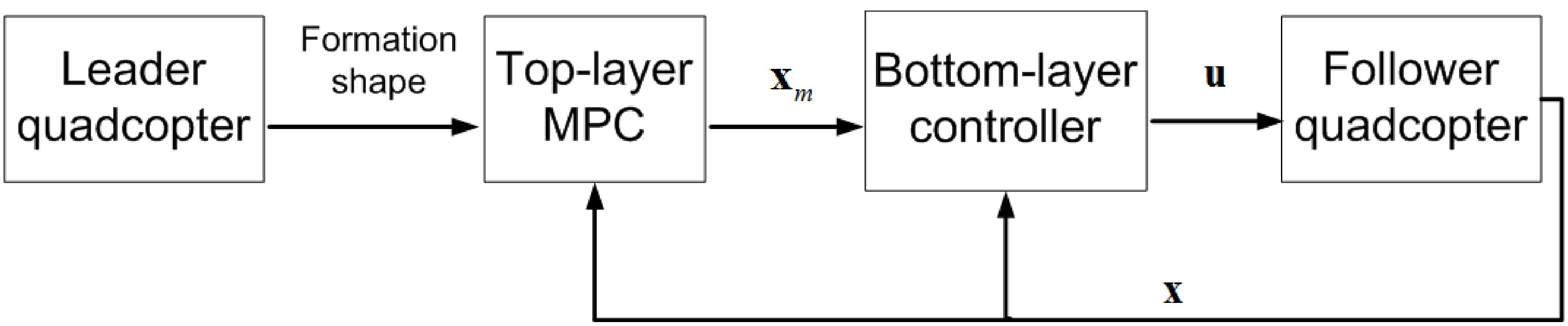
The Big Picture

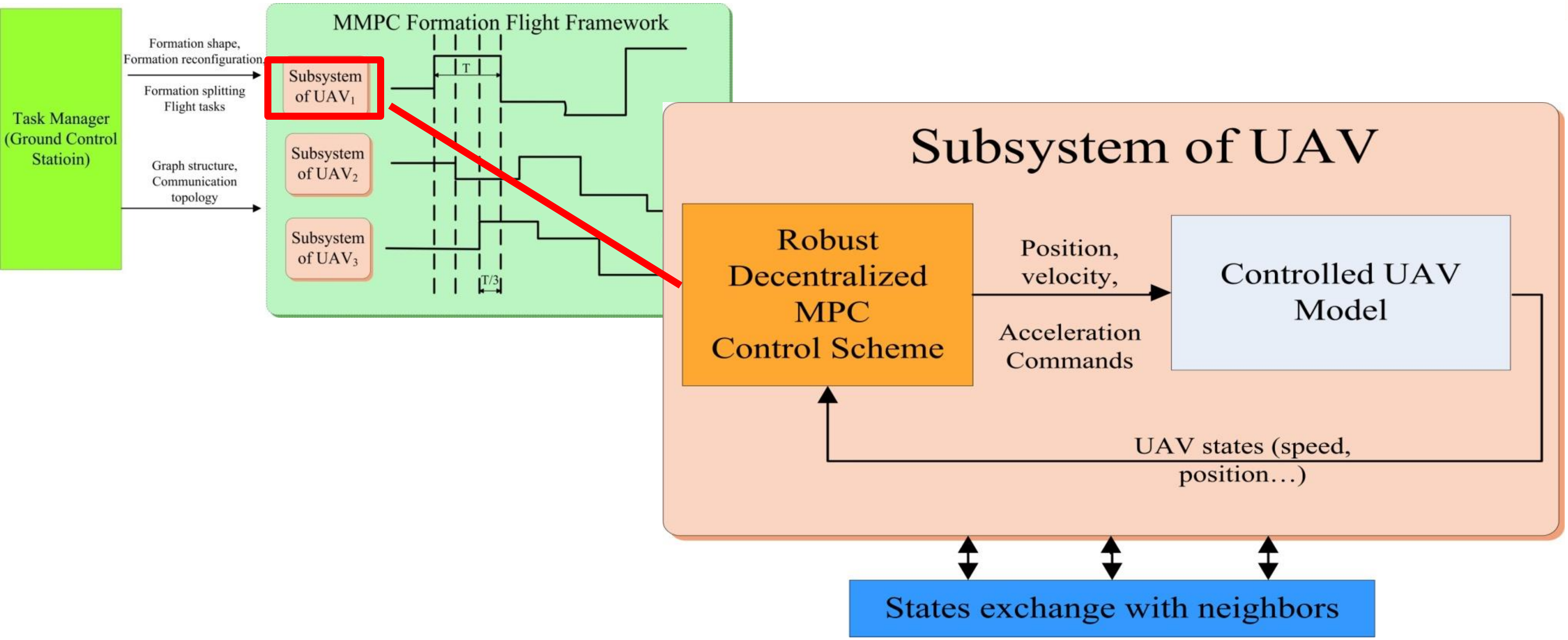


Two-Layer Formation Flight Framework

- The MPC controller generate the optimized collision-free state reference trajectory which satisfies all kind of constraints and robust to the input disturbances
  - two modifications, i.e. the control input hold and variable prediction horizon are made and combined to allow the real-time formation flight implementation
- Robust feedback linearization controller tracks the optimal state reference and suppress any tracking errors during the MPC update interval



Decentralized Formation Control Scheme



- Formation controller resides on top of the individual UAV autopilot
- Communication topology: each UAV only exchanges information with its neighbors

Formation Flight System under MMPC

$$\boldsymbol{x}_{k+1} = \begin{bmatrix} \boldsymbol{x}^1_{k+1} \\ \vdots \\ \boldsymbol{x}^m_{k+1} \end{bmatrix} = \begin{pmatrix} A^1 & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A^m \end{pmatrix} \begin{bmatrix} \boldsymbol{x}^1_k \\ \vdots \\ \boldsymbol{x}^m_k \end{bmatrix} + \begin{pmatrix} B^1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & B^m \end{pmatrix} \begin{bmatrix} \boldsymbol{u}^1_k \\ \vdots \\ \boldsymbol{u}^m_k \end{bmatrix}$$

$$= A\boldsymbol{x}_k + B\boldsymbol{u}_k$$

$$= A\boldsymbol{x}_k + \sum_{j=1}^{m} B_j \boldsymbol{u}_k^j$$

Under MMPC scheme, the whole formation system can be represented as a periodic linear system with one UAV's input at a time:

$$\boldsymbol{x}_{k+1} = A\boldsymbol{x}_k + B_{\sigma(k)}\boldsymbol{u}_k^{\sigma(k)} \qquad \sigma(k) = (k \bmod m) + 1$$

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Formation Flight System under MMPC

In order to make the whole linear periodic system closed-loop stable by using the MMPC scheme, the following two additional terms need to be added to the optimization problem formulation:

- Terminal cost term:

$$F\left(\boldsymbol{x}_{k+N}\right)=\min_{\boldsymbol{u}_k^{\sigma(k)}}\left\{\sum_{i=N}^{\infty}\left(\left\|\boldsymbol{x}_{k+i+1}-\boldsymbol{x}^d_{k+i+1}\right\|_Q^2+\left\|\boldsymbol{u}_{k+i}^{\sigma(k)}\right\|_R^2\right) \mid \boldsymbol{x}_{k+1}=A\boldsymbol{x}_k+B_{\sigma(k)}\boldsymbol{u}_k^{\sigma(k)}\right\}$$

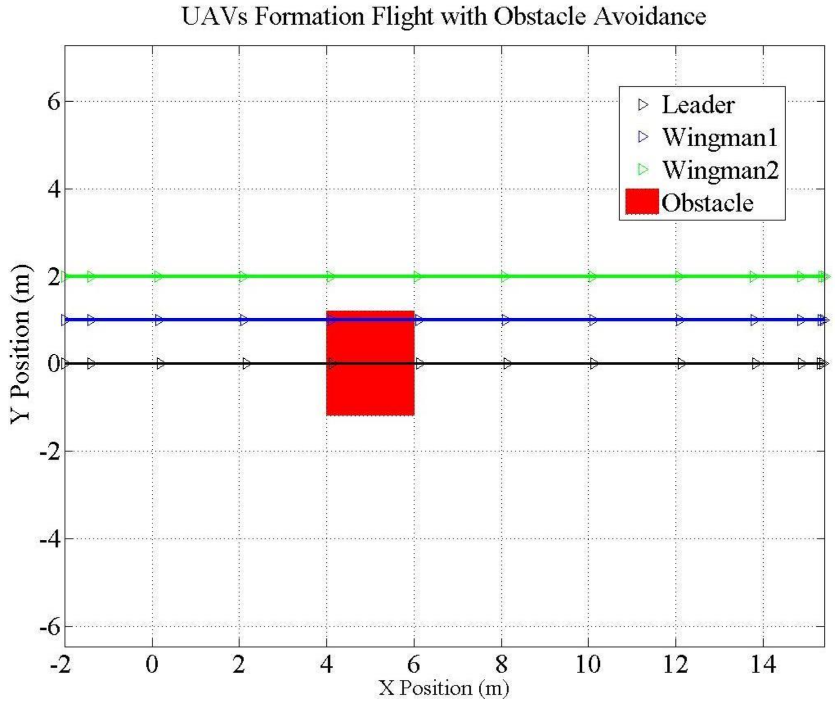
$$=\boldsymbol{x}^T_{k+N}\bar{P}_{\sigma(k+N)}\boldsymbol{x}_{k+N}+2\bar{q}^T_{\sigma(k+N)}\boldsymbol{x}_{k+N}+\bar{r}_{\sigma(k+N)}$$

- Terminal states constraints:

$$\boldsymbol{x}_{k+N+1}\in\mathcal{X}_I\left(K_{\sigma(k)}\right)$$

Collision Avoidance Extension

- In the current MPC collision avoidance scheme:
  - Size of obstacles usually not included in the constraint formulation
    - Trajectories may penetrate small obstacles
  - Lack of treatment for pop-up obstacles
    - Obstacles are assumed to be known a priori



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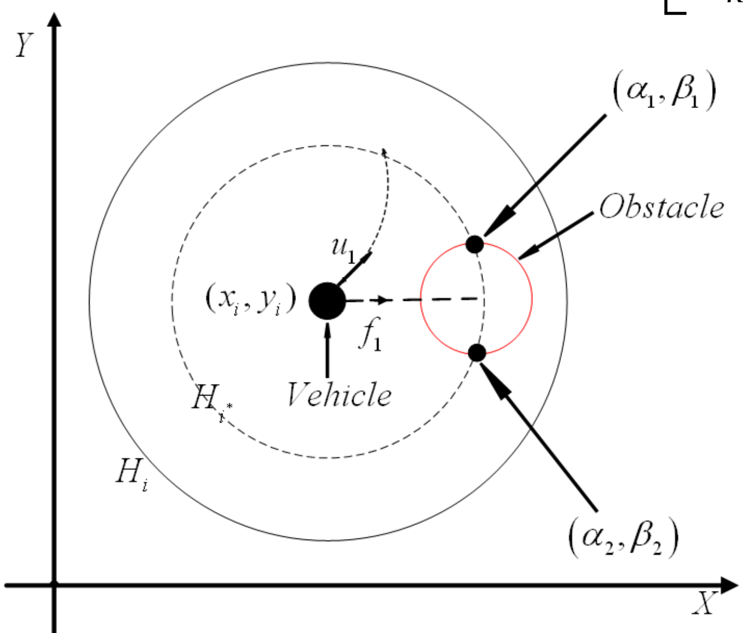
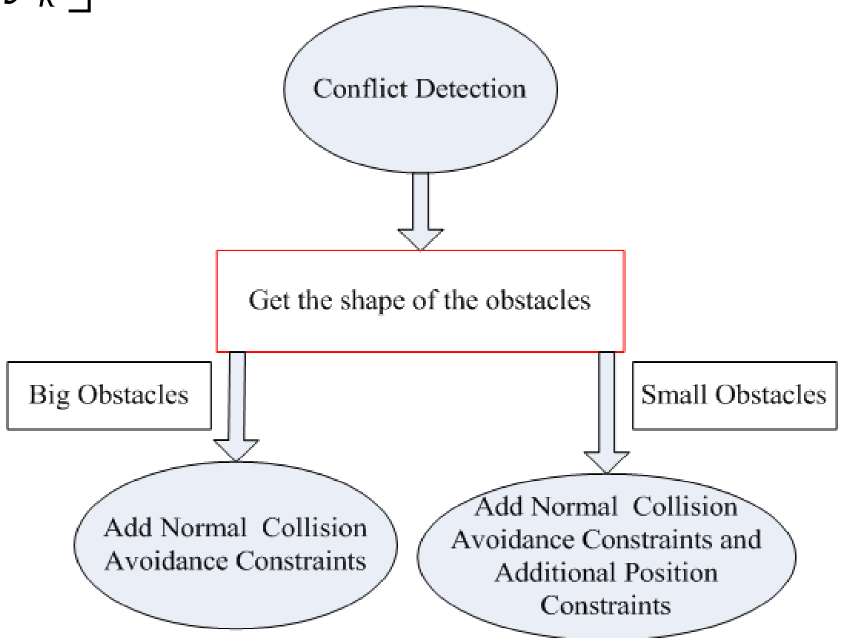
Collision Avoidance Modification

- Proposed improvement:
  - Combine the temporal (time) and spatial (sensor range) horizons
    - Add position constraint to prevent the next vehicle predicted position to penetrate the small obstacle

$$norm\left(\begin{bmatrix}\hat{x}_{k+1|k}\\ \hat{y}_{k+1|k}\end{bmatrix}-\begin{bmatrix}x_k\\ y_k\end{bmatrix}\right)\leq H_{i^*}$$





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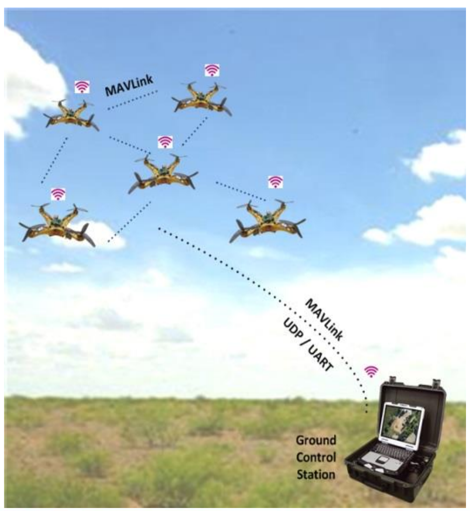
NTU Swarming Research
-- Test-bed Architecture

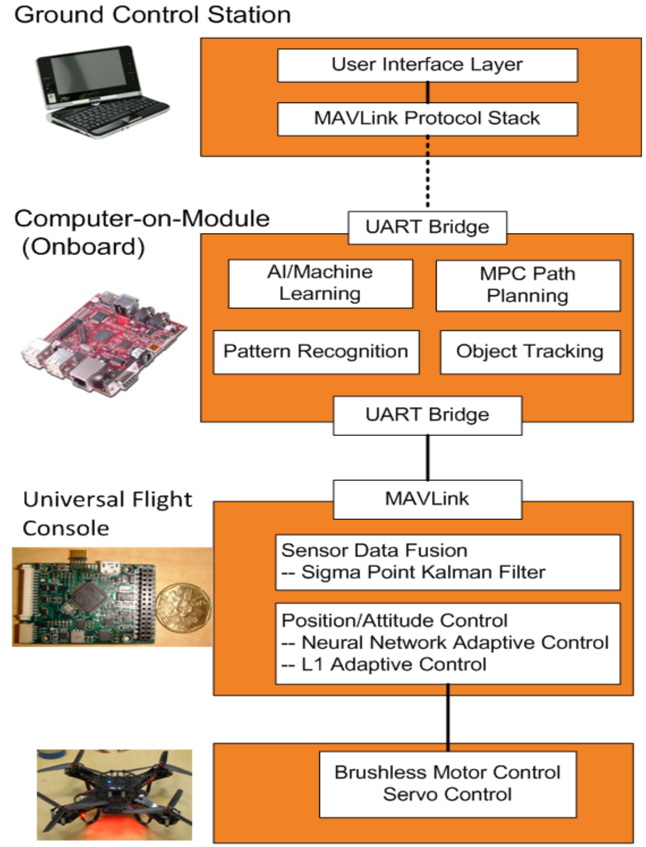


Design Based On:
- PIXHAWK of ETH Zurich
- Openpilot project

Design Philosophy:
- High performance
- Efficiency
- Ease of development
- Low cost

Ground Control Station:
- QGC
- Matlab/Simulink

Ideal For:
- Formation/swarm
- Vision-based navigation



UAV Formation Flight

NTU-built drones to fly at S'pore Airshow

[Overview of UAV Show Formation Flight]()

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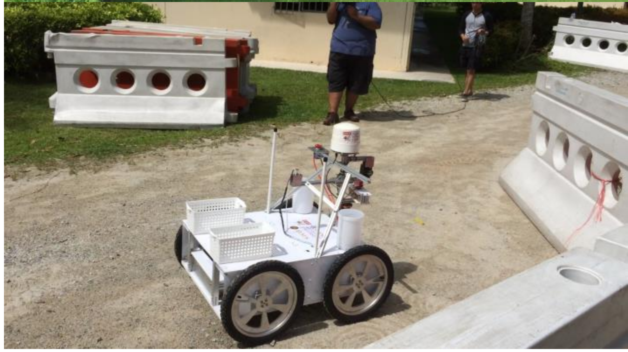
TechX Challenge 2013 Competition Scenarios

- Large outdoor arena
- Multiple indoor settings
- Rugged outdoor terrains
- Multiple indoor/outdoor stairs
- GPS challenged in many locations
- Multiple robot coordination
- Search targets

Tasks in TechX-Challenge 2013

1. Outdoor GPS based Navigation and obstacle avoidance
2. Indoor mapping and navigation
3. Outdoor exploration and target identification
4. Indoor exploration and target identification
5. Entrance and exit of buildings
6. Searching for staircase and climbing
7. Communication and report to Operation Control Unit
8. Integration and mission control



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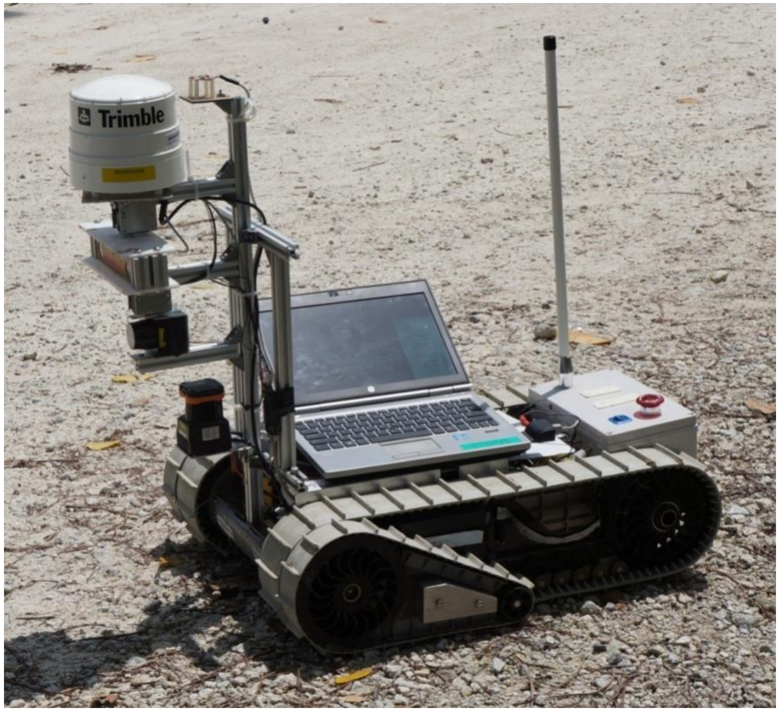
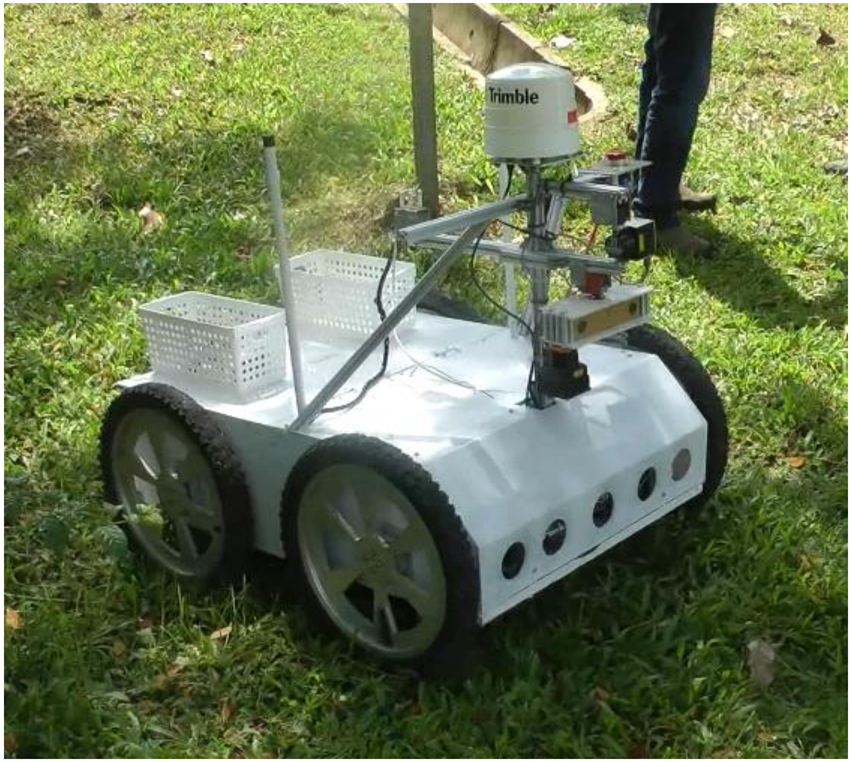
Tracked Robot



Robot Platforms



Wheeled Robots

Multiple configurations for
-- Speed to run in a large arena
-- Overcoming stairs

Video

Key Achievements:

Platform designs

In-house designs for urban and off-road environments with low cost sensor suites and communication between robots;

Outdoor mobility

GPS based outdoor navigation with obstacle avoidance in rugged terrains,

Indoor mobility

SLAM-based exploration and researching;

Stair climbing and descending
Laser based confirmation and climbing up and down;

Target identification and engagement
Stereo vision based identification and confirmation

Outdoor and indoor transitions

Vision/laser based navigation through building entrances.

Integrated missions
Functionalities have been successfully integrated.

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- **TechX Challenges 2013**

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Conclusion

- Distributed optimization and control has wide applications
- We studied a distributed optimal control problem based on a consensus approach
- Sufficient conditions were given for convergence to the optimal solutions
- We also discussed the discrete implementation and convergence rate
- Future work include different constraints for different agent.
- MMPC based formation flight was discussed and implemented
- Distributed MMPC will be investigated for outdoor UAVs

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Thank you !