

## Less conservative stability criteria for linear systems with interval time-varying delays

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### SUMMARY

The problem of the stability of a linear system with an interval time-varying delay is investigated. A new Lyapunov–Krasovskii functional that fully uses information about the lower bound of the time-varying delay is constructed to derive new stability criteria. It is proved that the proposed Lyapunov–Krasovskii functional can lead to less conservative results than some existing ones. Based on the proposed Lyapunov–Krasovskii functional, two stability conditions are developed using two different methods to estimate Lyapunov–Krasovskii functional's derivative. Two numerical examples are given to illustrate that the two stability conditions are complementary and yield a larger maximum upper bound of the time-varying delay than some existing results. Copyright © 2013 John Wiley & Sons, Ltd.

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### 1. INTRODUCTION

Consider the linear system with an interval time-varying delay described by

$$\begin{cases} \dot{x}(t) = Ax(t) + A_1x(t - d(t)), & t \geq 0 \\ x(\theta) = \phi(\theta), & \theta \in [-h_2, 0] \end{cases} \quad (1)$$

where  $x(t) \in \mathbb{R}^n$  is the state vector,  $A \in \mathbb{R}^{n \times n}$  and  $A_1 \in \mathbb{R}^{n \times n}$  are constant system matrices, the initial condition  $\phi(\theta)$  is a continuously differentiable vector-valued function, and  $d(t)$  is a time-varying differentiable function satisfying

$$0 < h_1 \leq d(t) \leq h_2, \quad \dot{d}(t) \leq \mu \quad (2)$$

where  $0 < h_1 \leq h_2$  and  $\mu$  are constants.

The problem of stability of the system described by (1)–(2) has received much attention in recent years, because it can model some systems such as networked control systems [1, 2]. Several techniques can be utilized to deal with the stability problem. For example, a discretized Lyapunov functional method was proposed in [3, 4]. This method is very effective, and a coarse discretization can yield satisfactory results. A descriptor transformation method was developed in [5], which is most effective among the four kind of model transformation methods in the literature. To further reduce conservatism of stability conditions, a free-weighting matrices method was proposed by He *et al.* [6, 7]. This method does not introduce any model transformations and bounding techniques for

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cross terms and can yield less conservative results than the descriptor transformation method. Jensen inequality technique [8, 9] was also used to bound the integral terms in the derivative of Lyapunov functional. A convex combination technique was developed in [9–11] to obtain some less conservative stability criteria for linear systems with time-varying delay. A delay fractioning scheme was put forward in [12] where the delay is divided into some sub-intervals and a new Lyapunov functional was established to derive less conservative results. Besides the results mentioned previously, some delay-dependent stability criteria have been reported in the literature [13–24].

As mentioned in our previous work [25], information about the lower bound of time-varying delay  $d(t)$  should be taken into consideration when constructing a Lyapunov–Krasovskii functional. As a consequence, the following Lyapunov–Krasovskii functional containing some information about the lower bound of delay was introduced in [25]

$$\begin{aligned}
 V(x_t) = & \rho^T(t)P\rho(t) + \int_{t-d(t)}^{t-h_1} x^T(s)Sx(s)ds \\
 & + \int_{t-h_1}^t \zeta^T(s)Q_1\zeta(s)ds + \int_{t-h_2}^{t-h_1} \zeta^T(s)Q_2\zeta(s)ds \\
 & + \int_{-h_1}^0 \int_{t+\theta}^t \dot{x}^T(s)Z_1\dot{x}(s)dsd\theta + \int_{-h_2}^{-h_1} \int_{t+\theta}^t \dot{x}^T(s)Z_2\dot{x}(s)dsd\theta \\
 & + \int_{-h_1}^0 \int_{t+\theta}^t x^T(s)Z_3x(s)dsd\theta + \int_{-h_2}^{-h_1} \int_{t+\theta}^t x^T(s)Z_4x(s)dsd\theta \\
 & + \int_{-h_1}^0 \int_{\theta}^0 \int_{t+\lambda}^t \dot{x}^T(s)R_1\dot{x}(s)dsd\lambda d\theta + \int_{-h_2}^{-h_1} \int_{\theta}^0 \int_{t+\lambda}^t \dot{x}^T(s)R_2\dot{x}(s)dsd\lambda d\theta
 \end{aligned} \tag{3}$$

where  $\rho(t) = col\{x(t), x(t - h_1), x(t - h_2), \int_{t-h_1}^t x(s)ds, \int_{t-h_2}^{t-h_1} x(s)ds\}$ ,  $\zeta(s) = col\{x(s), \dot{x}(s)\}$ . However, from the Lyapunov–Krasovskii functional (3), one can see clearly that there is no information about the lower bound of time-varying delay  $d(t)$  in the inner integral upper limits of the double integral terms and the triple integral term. Therefore, we think that Lyapunov–Krasovskii functional (3) does not sufficiently use the information about the lower bound of delay and thus may lead to conservative results. Based on this observation, the natural question is as follows: How can one improve the result in [25] by including more information about the lower bound of time-varying delay  $d(t)$ ? The answer to this question will enhance the stability analysis of the system described by (1)–(2). For this purpose, we construct the following Lyapunov–Krasovskii functional

$$\begin{aligned}
 V(x_t) = & \rho^T(t)P\rho(t) + \int_{t-d(t)}^{t-h_1} x^T(s)Sx(s)ds \\
 & + \int_{t-h_1}^t \zeta^T(s)Q_1\zeta(s)ds + \int_{t-h_2}^{t-h_1} \zeta^T(s)Q_2\zeta(s)ds \\
 & + \int_{-h_1}^0 \int_{t+\theta}^t \dot{x}^T(s)Z_1\dot{x}(s)dsd\theta + \int_{-h_2}^{-h_1} \int_{t+\theta}^{t-h_1} \dot{x}^T(s)Z_2\dot{x}(s)dsd\theta \\
 & + \int_{-h_1}^0 \int_{t+\theta}^t x^T(s)Z_3x(s)dsd\theta + \int_{-h_2}^{-h_1} \int_{t+\theta}^{t-h_1} x^T(s)Z_4x(s)dsd\theta \\
 & + \int_{-h_1}^0 \int_{\theta}^0 \int_{t+\lambda}^t \dot{x}^T(s)R_1\dot{x}(s)dsd\lambda d\theta + \int_{-h_2}^{-h_1} \int_{\theta}^{-h_1} \int_{t+\lambda}^{t-h_1} \dot{x}^T(s)R_2\dot{x}(s)dsd\lambda d\theta
 \end{aligned} \tag{4}$$

After comparing (3) with (4) carefully, one can see that the inner integral upper limits of  $\int_{-h_2}^{-h_1} \int_{t+\theta}^{t-h_1} \dot{x}^T(s)Z_2\dot{x}(s)dsd\theta$  and  $\int_{-h_2}^{-h_1} \int_{t+\theta}^{t-h_1} x^T(s)Z_4x(s)dsd\theta$  are  $t - h_1$ , while the inner integral upper limits of the triple integral term  $\int_{-h_2}^{-h_1} \int_{\theta}^{-h_1} \int_{t+\lambda}^{t-h_1} \dot{x}^T(s)R_2\dot{x}(s)dsd\lambda d\theta$  are  $-h_1$  and

$t - h_1$ . In this paper, we will prove that less conservative results can be obtained by employing Lyapunov–Krasovskii functional (4) than using (3).

Notice that there are some double-integral terms such as  $-\int_{-h_2}^{-h_1} \int_{t+\theta}^{t-h_1} \dot{x}^T(s) R_2 \dot{x}(s) ds d\theta$  in the derivative of the Lyapunov–Krasovskii functional due to the triple-integral terms in (4). How to deal with such terms appropriately is another problem we need to investigate. In this paper, two different methods are utilized to cope with this term. One method is to divide it into three parts, and then estimate these three parts, respectively. Another method is to introduce some free-weighting matrices and use a quadratic inequality to eliminate the double integral term. Two numerical examples illustrate that the above two methods yield complementary results.

## 2. MAIN RESULTS

Denote  $\langle \star \rangle_{sym} = (\star) + (\star)^T$ ,  $h_{12} = h_2 - h_1$ ,  $h_s = \frac{h_2^2 - h_1^2}{2}$ ,  $\beta_1 = d(t) - h_1$ ,  $\beta_2 = h_2 - d(t)$ ,  $\xi(t) = col\{x(t), x(t - d(t)), x(t - h_1), x(t - h_2), \dot{x}(t - h_1), \dot{x}(t - h_2), \int_{t-h_1}^t x(s) ds\}$  and  $e_i$  ( $i = 1, 2, \dots, 7$ ) are block entry matrices. For example,  $e_3^T = [0 \ 0 \ I \ 0 \ 0 \ 0 \ 0]$ . We now state and establish the following stability result for the system described by (1)–(2).

### Theorem 1

Given scalars  $h_1, h_2$ , and  $\mu$ , the system described by (1)–(2) is asymptotically stable if there exist matrices  $P > 0, Q_1 > 0, Q_2 > 0, S > 0, Z_j > 0, j = 1, \dots, 4, R_1 > 0, R_2 > 0$  and any matrices  $H_i, F_i, M_i$ , and  $N_i, i = 1, \dots, 4$  with appropriate dimensions such that

$$\Xi_1 = \begin{bmatrix} \Theta & \Upsilon^T - \hat{M} & \hat{F} & \hat{M} \\ * & -\frac{Z_4}{h_{12}} & 0 & 0 \\ * & * & -\frac{Z_2}{h_{12}} & 0 \\ * & * & * & -\frac{2R_2}{h_{12}^2} \end{bmatrix} < 0 \tag{5}$$

$$\Xi_2 = \begin{bmatrix} \Omega + \langle h_{12} \hat{N} e_3^T \rangle_{sym} & \Upsilon^T - \hat{N} & \hat{H} & \hat{N} \\ * & -\frac{Z_4}{h_{12}} & 0 & 0 \\ * & * & -\frac{Z_2}{h_{12}} & 0 \\ * & * & * & -\frac{2R_2}{h_{12}^2} \end{bmatrix} < 0 \tag{6}$$

where

$$\begin{aligned} \Theta &= \Omega + \langle h_{12} \hat{M} e_2^T \rangle_{sym} - (e_3 - e_2) R_2 (e_3^T - e_2^T) \\ \Omega &= \langle \Phi P \Psi^T + \hat{F} (e_2^T - e_4^T) + \hat{H} (e_3^T - e_2^T) \rangle_{sym} + A_c^T Y A_c + \Lambda \\ &\quad - [e_4 \ e_6] Q_2 [e_4 \ e_6]^T + [e_3 \ e_5] (Q_2 - Q_1) [e_3 \ e_5]^T \\ &\quad + [e_1 \ A_c] Q_1 [e_1 \ A_c]^T - (e_1 - e_3) \frac{Z_1}{h_1} (e_1^T - e_3^T) \\ &\quad - (h_1 e_1 - e_7) \frac{2R_1}{h_1^2} (h_1 e_1^T - e_7^T) \end{aligned}$$

$$\Phi = [e_1 \ e_3 \ e_4 \ e_7 \ 0]$$

$$\Psi = [A_c^T \ e_5 \ e_6 \ e_1 - e_3 \ e_3 - e_4]$$

$$\Lambda = \text{diag} \left\{ h_1 Z_3, -(1 - \mu)S, S + h_{12}Z_4, 0, h_{12}Z_2 + \frac{h_{12}^2}{2}R_2, 0, -\frac{Z_3}{h_1} \right\}$$

$$A_c = [A \ A_1 \ 0 \ 0 \ 0 \ 0 \ 0]$$

$$Y = h_1 Z_1 + \frac{h_1^2}{2}R_1$$

$$\Upsilon = [P_{15}^T A + P_{45}^T \ P_{15}^T A_1 \ -P_{45}^T + P_{55} \ -P_{55} \ P_{25}^T \ P_{35}^T \ 0]$$

$$\hat{N} = [N_1^T \ N_2^T \ N_3^T \ N_4^T \ 0 \ 0 \ 0]^T$$

$$\hat{M} = [M_1^T \ M_2^T \ M_3^T \ M_4^T \ 0 \ 0 \ 0]^T$$

$$\hat{F} = [F_1^T \ F_2^T \ F_3^T \ F_4^T \ 0 \ 0 \ 0]^T$$

$$\hat{H} = [H_1^T \ H_2^T \ H_3^T \ H_4^T \ 0 \ 0 \ 0]^T$$

*Proof*

Notice that similar to [6, 7], the following equations hold

$$\alpha_1 := 2\xi^T(t)\hat{H} \left[ x(t - h_1) - x(t - d(t)) - \int_{t-d(t)}^{t-h_1} \dot{x}(s)ds \right] = 0 \quad (7)$$

$$\alpha_2 := 2\xi^T(t)\hat{F} \left[ x(t - d(t)) - x(t - h_2) - \int_{t-h_2}^{t-d(t)} \dot{x}(s)ds \right] = 0 \quad (8)$$

$$\alpha_3 := 2\xi^T(t)\hat{N} \left[ \beta_1 x(t - h_1) - \int_{t-d(t)}^{t-h_1} x(s)ds - \int_{-d(t)}^{-h_1} \int_{t+\theta}^{t-h_1} \dot{x}(s)dsd\theta \right] = 0 \quad (9)$$

$$\alpha_4 := 2\xi^T(t)\hat{M} \left[ \beta_2 x(t - d(t)) - \int_{t-h_2}^{t-d(t)} x(s)ds - \int_{-h_2}^{-d(t)} \int_{t+\theta}^{t-d(t)} \dot{x}(s)dsd\theta \right] = 0 \quad (10)$$

Notice also that the following equation holds

$$\begin{aligned} - \int_{-h_2}^{-h_1} \int_{t+\theta}^{t-h_1} \dot{x}^T(s)R_2\dot{x}(s)dsd\theta &= - \int_{-h_2}^{-d(t)} \int_{t+\theta}^{t-d(t)} \dot{x}^T(s)R_2\dot{x}(s)dsd\theta \\ &\quad - \int_{-d(t)}^{-h_1} \int_{t+\theta}^{t-h_1} \dot{x}^T(s)R_2\dot{x}(s)dsd\theta - \beta_2 \int_{t-d(t)}^{t-h_1} \dot{x}^T(s)R_2\dot{x}(s)ds \end{aligned} \quad (11)$$

Taking the derivative of the Lyapunov–Krasovskii functional (4) along the trajectory of system (1) yields

$$\begin{aligned}
 \dot{V}(x_t) = & 2\rho^T(t)P\dot{\rho}(t) + \xi^T(t)Q_1\xi(t) + \xi^T(t-h_1)(Q_2 - Q_1)\xi(t-h_1) \\
 & - \xi^T(t-h_2)Q_2\xi(t-h_2) + x^T(t-h_1)(S + h_{12}Z_4)x(t-h_1) \\
 & + \dot{x}^T(t)Y\dot{x}(t) - (1 - \dot{d}(t))x^T(t-d(t))Sx(t-d(t)) \\
 & + \dot{x}^T(t-h_1)\left(h_{12}Z_2 + \frac{h_{12}^2}{2}R_2\right)\dot{x}(t-h_1) + h_1x^T(t)Z_3x(t) \\
 & - \int_{t-h_1}^t \dot{x}^T(s)Z_1\dot{x}(s)ds - \int_{t-h_2}^{t-d(t)} \dot{x}^T(s)Z_2\dot{x}(s)ds \\
 & - \int_{t-d(t)}^{t-h_1} \dot{x}^T(s)Z_2\dot{x}(s)ds - \int_{t-h_1}^t x^T(s)Z_3x(s)ds \\
 & - \int_{t-h_2}^{t-h_1} x^T(s)Z_4x(s)ds - \int_{-h_1}^0 \int_{t+\theta}^t \dot{x}^T(s)R_1\dot{x}(s)dsd\theta \\
 & - \int_{-h_2}^{-d(t)} \int_{t+\theta}^{t-d(t)} \dot{x}^T(s)R_2\dot{x}(s)dsd\theta - \int_{-d(t)}^{-h_1} \int_{t+\theta}^{t-h_1} \dot{x}^T(s)R_2\dot{x}(s)dsd\theta \\
 & - \beta_2 \int_{t-d(t)}^{t-h_1} \dot{x}^T(s)R_2\dot{x}(s)ds + \sum_{i=1}^4 \alpha_i
 \end{aligned} \tag{12}$$

Using Jensen inequality [8], one can obtain

$$- \int_{t-h_1}^t \dot{x}^T(s)Z_1\dot{x}(s)ds \leq -\xi^T(t)(e_1 - e_3)\frac{Z_1}{h_1}(e_1^T - e_3^T)\xi(t) \tag{13}$$

$$- \int_{t-h_1}^t x^T(s)Z_3x(s)ds \leq -\xi^T(t)e_7\frac{Z_3}{h_1}e_7^T\xi(t) \tag{14}$$

$$- \int_{-h_1}^0 \int_{t+\theta}^t \dot{x}^T(s)R_1\dot{x}(s)dsd\theta \leq -\xi^T(t)(h_1e_1 - e_7)\frac{2R_1}{h_1^2}(h_1e_1^T - e_7^T)\xi(t) \tag{15}$$

$$-\beta_2 \int_{t-d(t)}^{t-h_1} \dot{x}^T(s)R_2\dot{x}(s)ds \leq -\beta_2\xi^T(t)(e_3 - e_2)\frac{R_2}{h_{12}}(e_3^T - e_2^T)\xi(t) \tag{16}$$

Denoting  $E_1 = [0 \ 0 \ 0 \ 0 \ I]$ , it is easy to see that

$$2 \int_{t-h_2}^{t-h_1} x^T(s)ds E_1 P \dot{\rho}(t) = 2 \left[ \int_{t-d(t)}^{t-h_1} x^T(s)ds + \int_{t-h_2}^{t-d(t)} x^T(s)ds \right] E_1 P \dot{\rho}(t) \tag{17}$$

Clearly,

$$\begin{aligned}
 & 2 \int_{t-d(t)}^{t-h_1} x^T(s)ds E_1 P \dot{\rho}(t) - 2\xi^T(t)\hat{N} \int_{t-d(t)}^{t-h_1} x(s)ds \\
 & \leq \beta_1 \xi^T(t)(\Upsilon^T - \hat{N})Z_4^{-1}(\Upsilon - \hat{N}^T)\xi(t) + \int_{t-d(t)}^{t-h_1} x^T(s)Z_4x(s)ds
 \end{aligned} \tag{18}$$

$$\begin{aligned}
 & 2 \int_{t-h_2}^{t-d(t)} x^T(s)ds E_1 P \dot{\rho}(t) - 2\xi^T(t)\hat{M} \int_{t-h_2}^{t-d(t)} x(s)ds \\
 & \leq \beta_2 \xi^T(t)(\Upsilon^T - \hat{M})Z_4^{-1}(\Upsilon - \hat{M}^T)\xi(t) + \int_{t-h_2}^{t-d(t)} x^T(s)Z_4x(s)ds
 \end{aligned} \tag{19}$$

$$-2\xi^T(t)\hat{H}\int_{t-d(t)}^{t-h_1}\dot{x}(s)ds\leq\beta_1\xi^T(t)\hat{H}Z_2^{-1}\hat{H}^T\xi(t)+\int_{t-d(t)}^{t-h_1}\dot{x}^T(s)Z_2\dot{x}(s)ds\quad(20)$$

$$-2\xi^T(t)\hat{F}\int_{t-h_2}^{t-d(t)}\dot{x}(s)ds\leq\beta_2\xi^T(t)\hat{F}Z_2^{-1}\hat{F}^T\xi(t)+\int_{t-h_2}^{t-d(t)}\dot{x}^T(s)Z_2\dot{x}(s)ds\quad(21)$$

$$\begin{aligned} -2\xi^T(t)\hat{M}\int_{-h_2}^{-d(t)}\int_{t+\theta}^{t-d(t)}\dot{x}(s)dsd\theta\leq&\frac{\beta_2^2}{2}\xi^T(t)\hat{M}R_2^{-1}\hat{M}^T\xi(t) \\ &+\int_{-h_2}^{-d(t)}\int_{t+\theta}^{t-d(t)}\dot{x}^T(s)R_2\dot{x}(s)dsd\theta \end{aligned}\quad(22)$$

$$\begin{aligned} -2\xi^T(t)\hat{N}\int_{-d(t)}^{-h_1}\int_{t+\theta}^{t-h_1}\dot{x}(s)dsd\theta\leq&\frac{\beta_1^2}{2}\xi^T(t)\hat{N}R_2^{-1}\hat{N}^T\xi(t) \\ &+\int_{-d(t)}^{-h_1}\int_{t+\theta}^{t-h_1}\dot{x}^T(s)R_2\dot{x}(s)dsd\theta \end{aligned}\quad(23)$$

From (12)–(23), one can obtain

$$\dot{V}(x_t)\leq\xi^T(t)\Omega_{d(t)}\xi(t)\quad(24)$$

where  $\Omega_{d(t)}=\Omega+\langle\beta_2\hat{M}e_2^T+\beta_1\hat{N}e_3^T\rangle_{sym}-\frac{\beta_2}{h_{12}}(e_3-e_2)R_2(e_3^T-e_2^T)+\beta_1((\Upsilon^T-\hat{N})Z_4^{-1}(\Upsilon-\hat{N}^T)+\hat{H}Z_2^{-1}\hat{H}^T)+\beta_2((\Upsilon^T-\hat{M})Z_4^{-1}(\Upsilon-\hat{M}^T)+\hat{F}Z_2^{-1}\hat{F}^T)+\frac{\beta_2^2}{2}\hat{M}R_2^{-1}\hat{M}^T+\frac{\beta_1^2}{2}\hat{N}R_2^{-1}\hat{N}^T$ . Because its second order derivative with respect to  $d(t)$  is  $\hat{M}R_2^{-1}\hat{M}^T+\hat{N}R_2^{-1}\hat{N}^T\geq 0$ ,  $\xi^T(t)\Omega_{d(t)}\xi(t)$  is a convex quadratic function on  $d(t)$ . Therefore,  $\xi^T(t)\Omega_{d(t)}\xi(t)<0$  is equivalent to

$$\Omega_{d(t)}|_{d(t)=h_1}<0,\quad\Omega_{d(t)}|_{d(t)=h_2}<0\quad(25)$$

Using Schur complements, (25) is equivalent to (5)–(6). Therefore, if (5)–(6) are satisfied, then system described by (1)–(2) is asymptotically stable.  $\square$

From (11), one can see that  $-\int_{-h_2}^{-h_1}\int_{t+\theta}^{t-h_1}\dot{x}^T(s)R_2\dot{x}(s)dsd\theta$  is divided into three parts. In what follows, we use another method to deal with this term. It is clear to see that the following equation holds

$$0=2\xi^T(t)G\left[h_{12}x(t-h_1)-\int_{t-h_2}^{t-d(t)}x(s)ds-\int_{t-d(t)}^{t-h_1}x(s)ds-\int_{-h_2}^{-h_1}\int_{t+\theta}^{t-h_1}\dot{x}(s)dsd\theta\right]$$

Use the following inequality

$$\begin{aligned} -2\xi^T(t)G\int_{-h_2}^{-h_1}\int_{t+\theta}^{t-h_1}\dot{x}(s)dsd\theta\leq&\frac{h_{12}^2}{2}\xi^T(t)GR_2^{-1}G^T\xi(t) \\ &+\int_{-h_2}^{-h_1}\int_{t+\theta}^{t-h_1}\dot{x}^T(s)R_2\dot{x}(s)dsd\theta \end{aligned}$$

and then  $-\int_{-h_2}^{-h_1}\int_{t+\theta}^{t-h_1}\dot{x}^T(s)R_2\dot{x}(s)dsd\theta$  is eliminated from the derivative of the Lyapunov–Krasovskii functional. Following the similar line as the proof of Theorem 1, we arrive at the following result.

*Theorem 2*

Given scalars  $h_1, h_2$ , and  $\mu$ , the system described by (1)–(2) is asymptotically stable if there exist matrices  $P > 0, Q_1 > 0, Q_2 > 0, S > 0, Z_j > 0, j = 1, \dots, 4, R_1 > 0, R_2 > 0$  and any matrices  $J_i, L_i$ , and  $G_i, i = 1, \dots, 4$  with appropriate dimensions such that

$$\tilde{\Xi}_1 = \begin{bmatrix} \tilde{\Omega} & \Upsilon^T - \hat{G} & \hat{L} & \hat{G} \\ * & -\frac{Z_4}{h_{12}} & 0 & 0 \\ * & * & -\frac{Z_2}{h_{12}} & 0 \\ * & * & * & -\frac{2R_2}{h_{12}^2} \end{bmatrix} < 0 \tag{26}$$

$$\tilde{\Xi}_2 = \begin{bmatrix} \tilde{\Omega} & \Upsilon^T - \hat{G} & \hat{J} & \hat{G} \\ * & -\frac{Z_4}{h_{12}} & 0 & 0 \\ * & * & -\frac{Z_2}{h_{12}} & 0 \\ * & * & * & -\frac{2R_2}{h_{12}^2} \end{bmatrix} < 0 \tag{27}$$

where

$$\begin{aligned} \tilde{\Omega} &= \left\langle \Phi P \Psi^T + \hat{L} (e_2^T - e_4^T) + \hat{J} (e_3^T - e_2^T) + h_{12} \hat{G} e_3^T \right\rangle_{sym} \\ &\quad - [e_4 \ e_6] Q_2 [e_4 \ e_6]^T + [e_3 \ e_5] (Q_2 - Q_1) [e_3 \ e_5]^T \\ &\quad + [e_1 \ A_c] Q_1 [e_1 \ A_c]^T - (e_1 - e_3) \frac{Z_1}{h_1} (e_1^T - e_3^T) \\ &\quad - (h_1 e_1 - e_7) \frac{2R_1}{h_1^2} (h_1 e_1^T - e_7^T) + A_c^T Y A_c + \Lambda \\ \hat{G} &= [G_1^T \ G_2^T \ G_3^T \ G_4^T \ 0 \ 0 \ 0]^T \\ \hat{L} &= [L_1^T \ L_2^T \ L_3^T \ L_4^T \ 0 \ 0 \ 0]^T \\ \hat{J} &= [J_1^T \ J_2^T \ J_3^T \ J_4^T \ 0 \ 0 \ 0]^T \end{aligned}$$

and the other symbols are the same as those in Theorem 1.

*Remark 1*

To estimate bounds of  $-\int_{t-h_2}^{t-h_1} \dot{x}^T(s) Z_2 \dot{x}(s) ds$  and  $-\int_{t-h_2}^{t-h_1} x^T(s) Z_4 x(s) ds$  in (12), Jensen inequality was used in [25]. However, the method in [25] enlarges  $-\frac{d(t)-h_1}{h_2-d(t)}$  as  $-\frac{d(t)-h_1}{h_{12}}$ , which may introduce some conservatism as reported in [10]. Therefore, the free-weighting matrices method and the property of quadratic convex function are used to cope with these terms in this paper. While the other terms in (12) such as  $-\int_{t-h_1}^t \dot{x}^T(s) Z_1 \dot{x}(s) ds, -\int_{t-h_1}^t x^T(s) Z_3 x(s) ds$  and  $-\int_{-h_1}^0 \int_{t+\theta}^t \dot{x}^T(s) R_1 \dot{x}(s) ds d\theta$  are dealt with Jensen inequality. This is because such terms only involve information about the lower bound of time-varying delay  $d(t)$  for such terms Jensen inequality method gives the same level performance as free-weighting matrices method but with a small number of decision variables.

Based on the Lyapunov–Krasovskii functional (4), two delay-dependent stability criteria are obtained in Theorems 1 and 2. Compared with Lyapunov–Krasovskii functional (3), Lyapunov–Krasovskii functional (4) uses information about the lower bound of time-varying delay  $d(t)$  more sufficiently. In the following, we will prove that employing (4) one can derive a less conservative result than using (3). To show this, similar to the proof of Theorem 1, we have the following result by Lyapunov–Krasovskii functional (3).

*Theorem 3*

Given scalars  $h_1, h_2$ , and  $\mu$ , the system described by (1)–(2) is asymptotically stable if there exist matrices  $\bar{P} > 0, \bar{Q}_1 > 0, \bar{Q}_2 > 0, \bar{S} > 0, \bar{Z}_j > 0, j = 1, \dots, 4, \bar{R}_1 > 0, \bar{R}_2 > 0$  and any matrices  $\bar{H}_i, \bar{F}_i, \bar{M}_i$ , and  $\bar{N}_i, i = 1, \dots, 4$  with appropriate dimensions such that

$$\bar{\Theta}_1 = \begin{bmatrix} \bar{\Theta} & \bar{\Upsilon}^T - \bar{M} & \bar{F} & \bar{M} \\ * & -\frac{\bar{Z}_4}{h_{12}} & 0 & 0 \\ * & * & -\frac{\bar{Z}_2}{h_{12}} & 0 \\ * & * & * & -\frac{2\bar{R}_2}{h_{12}^2} \end{bmatrix} < 0 \tag{28}$$

$$\bar{\Theta}_2 = \begin{bmatrix} \bar{\Omega} + \langle h_{12}\bar{N}e_2 \rangle_{sym} & \bar{\Upsilon}^T - \bar{N} & \bar{H} & \bar{N} \\ * & -\frac{\bar{Z}_4}{h_{12}} & 0 & 0 \\ * & * & -\frac{\bar{Z}_2}{h_{12}} & 0 \\ * & * & * & -\frac{2\bar{R}_2}{h_{12}^2} \end{bmatrix} < 0 \tag{29}$$

where

$$\begin{aligned} \bar{\Theta} &= \bar{\Omega} + \langle h_{12}\bar{M}e_2^T \rangle_{sym} - (e_3 - e_2) \bar{R}_2 (e_3^T - e_2^T) \\ \bar{\Omega} &= \langle \Phi \bar{P} \Psi^T + \bar{F} (e_2^T - e_4^T) + \bar{H} (e_3^T - e_2^T) \rangle_{sym} + A_c^T \bar{Y} A_c + \bar{\Lambda} \\ &\quad - [e_4 \ e_6] \bar{Q}_2 [e_4 \ e_6]^T + [e_3 \ e_5] (\bar{Q}_2 - \bar{Q}_1) [e_3 \ e_5]^T \\ &\quad + [e_1 \ A_c] \bar{Q}_1 [e_1 \ A_c]^T - (e_1 - e_3) \frac{\bar{Z}_1 + h_{12}\bar{R}_2}{h_1} (e_1^T - e_3^T) \\ &\quad - (h_1 e_1 - e_7) \frac{2\bar{R}_1}{h_1^2} (h_1 e_1^T - e_7^T) \\ \bar{\Lambda} &= \text{diag} \left\{ h_1 \bar{Z}_3 + h_{12} \bar{Z}_4, -(1 - \mu) \bar{S}, \bar{S}, 0, 0, 0, -\frac{\bar{Z}_3}{h_1} \right\} \\ \bar{Y} &= h_1 \bar{Z}_1 + \frac{h_1^2}{2} \bar{R}_1 + h_{12} \bar{Z}_2 + h_s \bar{R}_2 \\ \bar{\Upsilon} &= [\bar{P}_{15}^T A + \bar{P}_{45}^T \ \bar{P}_{15}^T A_1 \ -\bar{P}_{45}^T + \bar{P}_{55} \ -\bar{P}_{55} \ \bar{P}_{25}^T \ \bar{P}_{35}^T \ 0] \\ \bar{N} &= [\bar{N}_1^T \ \bar{N}_2^T \ \bar{N}_3^T \ \bar{N}_4^T \ 0 \ 0 \ 0]^T \\ \bar{M} &= [\bar{M}_1^T \ \bar{M}_2^T \ \bar{M}_3^T \ \bar{M}_4^T \ 0 \ 0 \ 0]^T \\ \bar{F} &= [\bar{F}_1^T \ \bar{F}_2^T \ \bar{F}_3^T \ \bar{F}_4^T \ 0 \ 0 \ 0]^T \\ \bar{H} &= [\bar{H}_1^T \ \bar{H}_2^T \ \bar{H}_3^T \ \bar{H}_4^T \ 0 \ 0 \ 0]^T \end{aligned}$$

and the other symbols are same as those in Theorem 1.

The relationship between Theorems 1 and 3 is established as the following theorem.

*Theorem 4*

Consider the system described by (1)–(2). Given scalars  $h_1, h_2$ , and  $\mu$ , if there exist matrices  $\bar{P} > 0, \bar{Q}_1 > 0, \bar{Q}_2 > 0, \bar{S} > 0, \bar{Z}_j > 0, j = 1, \dots, 4, \bar{R}_1 > 0, \bar{R}_2 > 0$  and any matrices  $\bar{H}_i, \bar{F}_i, \bar{M}_i$  and  $\bar{N}_i, i = 1, \dots, 4$  with appropriate dimensions such that (28)–(29) hold, then  $P = \bar{P} > 0, Q_1 = \bar{Q}_1 + \text{diag}\{h_{12}\bar{Z}_4, h_{12}\bar{Z}_2 + \frac{h_{12}^2}{2}\bar{R}_2\} > 0, Q_2 = \bar{Q}_2 > 0, S = \bar{S} > 0, Z_1 = \bar{Z}_1 + h_{12}\bar{R}_2 > 0, Z_j = \bar{Z}_j > 0, j = 2, \dots, 4, R_1 = \bar{R}_1 > 0, R_2 = \bar{R}_2 > 0, H_i = \bar{H}_i, F_i = \bar{F}_i, M_i = \bar{M}_i$  and  $N_i = \bar{N}_i, i = 1, \dots, 4$  are feasible solutions to (5)–(6).



*Proof*

Suppose matrices  $\bar{P} > 0, \bar{Q}_1 > 0, \bar{Q}_2 > 0, \bar{S} > 0, \bar{Z}_j > 0, j = 1, \dots, 4, \bar{R}_1 > 0, \bar{R}_2 > 0, \bar{H}_i, \bar{F}_i, \bar{M}_i,$  and  $\bar{N}_i, i = 1, \dots, 4$  are feasible solutions to (28)–(29). Define  $P = \bar{P}, Q_1 = \bar{Q}_1 + \text{diag}\{h_{12}\bar{Z}_4, h_{12}\bar{Z}_2 + \frac{h_{12}^2}{2}\bar{R}_2\}, Q_2 = \bar{Q}_2, S = \bar{S}, Z_1 = \bar{Z}_1 + h_{12}\bar{R}_2, Z_j = \bar{Z}_j, j = 2, \dots, 4, R_1 = \bar{R}_1, R_2 = \bar{R}_2, H_i = \bar{H}_i, F_i = \bar{F}_i, M_i = \bar{M}_i$  and  $N_i = \bar{N}_i, i = 1, \dots, 4$ . Substitute  $\bar{P} = P, \bar{Q}_1 = Q_1 - \text{diag}\{h_{12}Z_4, h_{12}Z_2 + \frac{h_{12}^2}{2}R_2\}, \bar{Q}_2 = Q_2, \bar{S} = S, \bar{Z}_1 = Z_1 - h_{12}R_2, \bar{Z}_j = Z_j, j = 2, \dots, 4, \bar{R}_1 = R_1, \bar{R}_2 = R_2, \bar{H}_i = H_i, \bar{F}_i = F_i, \bar{M}_i = M_i$  and  $\bar{N}_i = N_i, i = 1, \dots, 4$  into  $\bar{\Xi}_1$  and  $\bar{\Xi}_2$ , and  $\Xi_1$  and  $\Xi_2$  will be obtained, respectively. Because  $\bar{\Xi}_1 < 0$  and  $\bar{\Xi}_2 < 0$ , then  $\Xi_1 < 0$  and  $\Xi_2 < 0$ , that is, (5)–(6) hold. Therefore,  $P, Q_1, Q_2, S, Z_j, j = 1, \dots, 4, R_1, R_2, H_i, F_i, M_i,$  and  $N_i, i = 1, \dots, 4$  are feasible solutions to (5)–(6). This completes the proof.  $\square$

*Remark 2*

From Theorem 4, it is clear that if there is a feasible solution to (28)–(29), then there must exist a feasible solution to (5)–(6), but not vice versa. In the next section, two numerical examples are given to confirm that Theorem 1 is less conservative than Theorem 3.

*Remark 3*

If choosing Lyapunov–Krasovskii functional (3) and following the similar line as Theorem 2, a corresponding result can be obtained. Using the similar method as in Theorem 4, it can be proved that Theorem 2 is less conservative than this corresponding result.

### 3. NUMERICAL EXAMPLES

In this section, two numerical examples are given to illustrate the effectiveness of the proposed method, that is, the method in this paper can yield less conservative results than some existing ones.

*Example 1*

Consider the following system [7] with

$$A = \begin{bmatrix} -0.5 & -2 \\ 1 & -1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -0.5 & -1 \\ 0 & 0.6 \end{bmatrix}$$

For various  $\mu$ , the maximum upper bounds of delay (MUBDs) for given lower bounds compared with those in [10, 21, 25] are listed in Table I. It is easy to see that the MUBDs obtained in this paper are much larger than those in [10, 21, 25]. From Table I, it also can be seen that Theorem 1 yields larger MUBDs than Theorem 3, which illustrates that Lyapunov–Krasovskii functional proposed in this paper can lead to less conservative results than that in [25]. From Table I, one can also see that Theorems 1 and 2 are complementary to each other. For example, when  $\mu = 0.15$  and  $h_1 = 0.5$ , Theorem 1 gives a larger MUBD than Theorem 2, while Theorem 2 gives a larger MUBD than Theorem 1 for  $\mu = 0.3$  and  $h_1 = 0.5$ .

*Example 2*

Consider the following system with

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Given different lower bounds, our objective is to calculate MUBDs, which keep the aforementioned system asymptotically stable. Table II lists the results for various  $\mu$  comparing those obtained in [10, 25]. It can be seen that the results obtained in this paper are better than those in [10, 25]. From Table II, we also can see that Theorem 1 is less conservative than Theorem 3, and Theorems 1 and 2 are complementary to each other.

Table I. Maximum upper bounds of delay for given  $h_1$  and different  $\mu$  for Example 1.

$h_1$	Methods	$\mu = 0.15$	$\mu = 0.3$	$\mu = 0.45$
0.5	[21] ( $N = 1$ )	1.4670	1.2874	1.1477
	[21] ( $N = 2$ )	1.4742	1.2889	1.1477
	[10]	1.8679	1.5746	1.3140
	[25]	1.9566	1.7129	1.4775
	Theorem 3	2.1300	1.8240	1.5360
	Theorem 2	2.1835	1.9108	1.5505
	Theorem 1	2.2173	1.8966	1.5447
0.8	[21] ( $N = 1$ )	1.4694	1.2882	1.1698
	[21] ( $N = 2$ )	1.4799	1.2901	1.1836
	[10]	1.9709	1.5978	1.3237
	[25]	2.2138	1.9001	1.6028
	Theorem 3	2.3200	2.0480	1.6670
	Theorem 2	2.4614	2.1571	1.6831
	Theorem 1	2.4753	2.1567	1.6812
1.1	[21] ( $N = 1$ )	1.4758	1.3108	1.3108
	[21] ( $N = 2$ )	1.4941	1.3222	1.3222
	[10]	1.9631	1.4598	1.3842
	[25]	2.4102	2.0136	1.6590
	Theorem 3	2.6150	2.3280	1.7790
	Theorem 2	2.7033	2.3797	1.7823
	Theorem 1	2.7229	2.3913	1.7831

Table II. Maximum upper bounds of delay for given  $h_1$  and different  $\mu$  for Example 2.

$h_1$	Methods	$\mu = 0.1$	$\mu = 0.3$	$\mu = 0.5$
0.6	[10]	—	—	—
	[25]	1.1632	1.1632	1.1632
	Theorem 3	1.3630	1.3630	1.3630
	Theorem 2	1.4810	1.4800	1.4800
	Theorem 1	1.4473	1.4473	1.4473
	0.8	[10]	—	—
[25]		1.3143	1.3143	1.3143
Theorem 3		1.4710	1.4710	1.4710
Theorem 2		1.5884	1.5827	1.5805
Theorem 1		1.5845	1.5783	1.5750
1.0		[10]	—	—
	[25]	1.4243	1.4243	1.4243
	Theorem 3	1.5190	1.5150	1.5130
	Theorem 2	1.6149	1.6144	1.6144
	Theorem 1	1.6154	1.6130	1.6118

#### 4. CONCLUSION

The problem of asymptotic stability for linear systems with interval time-varying delays has been investigated. A new type of Lyapunov–Krasovskii functional has been introduced, and it is proved that this Lyapunov–Krasovskii functional can yield less conservative results. Two complementary stability criteria have been obtained using two different methods to estimate the derivative of the Lyapunov functional. Two numerical examples have illustrated that results obtained in this paper are less conservative than some existing ones.

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## REFERENCES

1. Yue D, Han QL, Lam J. Network-based robust  $H_\infty$  control of systems with uncertainty. *Automatica* 2005; **41**:999–1007.
2. Zhang XM, Han QL. Novel delay-derivative-dependent stability criteria using new bounding techniques. *International Journal of Robust and Nonlinear Control* 2013; **23**:1419–1432.
3. Gu K. An improved stability criterion for systems with distributed delays. *International Journal of Robust and Nonlinear Control* 2003; **13**:819–831.
4. Gu K, Kharitonov VL, Chen J. *Stability of Time-Delay Systems*. Birkhauser: Boston, 2003.
5. Fridman E, Shaked U. An improved stabilization method for linear systems with time-delay. *IEEE Transactions on Automatic Control* 2002; **47**:1931–1937.
6. He Y, Wang QG, Lin C, Wu M. Delay-range-dependent stability for systems with time-varying delay. *Automatica* 2007; **43**:371–376.
7. He Y, Wang QG, Xie L, Lin C. Further improvement of free-weighting matrices technique for systems with time-varying delay. *IEEE Transactions on Automatic Control* 2007; **52**:293–299.
8. Gu K. Integral inequality in the stability problem of time-delay systems. *Proceedings of the 39th IEEE Conference on Decision and Control*, Sydney, Australia, 2000; 2805–2810.
9. Shao H. New delay-dependent stability criteria for systems with interval delay. *Automatica* 2009; **54**:744–749.
10. Zhang XM, Han QL. Stability of linear systems with interval time-varying delays arising from networked control systems. *Proceedings of the 36th Annual Conference on IEEE Industrial Electronics Society*, Glendale, USA, 2010; 225–230.
11. Park P, Ko JW. Stability and robust stability for systems with a time-varying delay. *Automatica* 2007; **43**:1855–1858.
12. Gouaisbaut F, Peaucelle D. Delay-dependent robust stability of time delay systems. *Proceedings of IFAC Symposium on Robust Control Design*, Toulouse, France, 2006; 453–458.
13. Niculescu SI. *Delay Effects on Stability: A robust control approach, Lecture Notes in Control and Information Sciences*. Springer-Verlag: London, 2001.
14. Park PG, Ko JW, Jeong C. Reciprocally convex approach to stability of systems with time-varying delays. *Automatica* 2007; **47**:235–238.
15. Sun J, Liu GP, Chen J. Delay-dependent stability and stabilization of neutral time-delay systems. *International Journal of Robust and Nonlinear Control* 2009; **19**:1364–1375.
16. Sun J, Liu GP, Chen J. Improved stability criteria for linear systems with time-varying delay. *IET Control Theory And Applications* 2010; **4**:683–689.
17. Ariba Y, Gouaisbaut F, Johansson KH. Stability interval for time-varying delay systems. *Proceedings of the 49th IEEE Conference on Decision and Control*, Atlanta, USA, 2010; 1017–1022.
18. Wu M, Feng ZY, He Y, She JH. Improved delay-dependent absolute stability and robust stability for a class of nonlinear systems with a time-varying delay. *International Journal of Robust and Nonlinear Control* 2010; **20**:694–702.
19. Ko JW, Jung S, Park P. Further enhancement of delay-range-dependent stability criteria for systems with time-varying delays. *International Conference on Control, Automation and Systems*, Gyeonggi-do, Korea, 2011; 1828–1832.
20. Kim JH. Note on stability of linear systems with time-varying delay. *Automatica* 2011; **47**:2118–2121.
21. Tang M, Wang YW, Wen C. Improved delay-range-dependent stability criteria for linear systems with interval time-varying delays. *IET Control Theory And Applications* 2012; **6**:868–873.
22. Sun XM, Liu GP, Rees D, Wang W. Delay-dependent stability for discrete systems with large delay sequence based on switching techniques. *Automatica* 2008; **44**:2902–2908.
23. Sun XM, Liu GP, Rees D, Wang W. Stability of systems with controller failure and time-varying delay. *IEEE Transactions on Automatic Control* 2008; **53**:2391–2396.
24. Zhu XL, Wang Y, Yang GH. New stability criteria for continuous-time systems with interval time-varying delay. *IET Control Theory And Applications* 2010; **4**:1101–1107.
25. Sun J, Liu GP, Chen J, Rees D. Improved delay-range-dependent stability criteria for linear systems with time-varying delays. *Automatica* 2010; **46**:466–470.