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Brief Paper

Distributed observer-based coordination for multiple Lagrangian systems using only position measurements

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Abstract: This study addresses the distributed coordination problem for multiple Lagrangian systems under a directed graph. Two cases are considered, namely, the distributed tracking control problem with a dynamic leader and the leaderless synchronisation problem. To overcome the difficulty that only positions are measured, a kind of new distributed observer is designed to estimate the velocity for each follower. The velocity observer is updated using only position information from the agent itself and from its neighbours. Based on the outputs of the observer, the distributed control protocols are proposed, respectively, such that the tracking errors locally exponentially converge to zero in the tracking scenario and the agents synchronise in the leaderless synchronisation scenario. Furthermore, the stability of the overall scheme is discussed under the directed interaction graphs that contain a directed spanning tree. Finally, cooperative simulations are provided to show the effectiveness of the proposed observer and control algorithms.

1 Introduction

Distributed cooperative control of multi-agent systems has received considerable attention from various scientific communities in recent years because of the broad applications, such as autonomous underwater vehicles, sensor networks and unmanned aerial vehicles [1–5]. To the best of our knowledge, cooperative control of linear systems described by single/double integrators are mainly focused on in most existing results [6–8]. Many issues such as consensus with non-uniform time-varying delays [9], model reference control with uncertain dynamics[10], tracking with Lipschitz-type dynamics [11], circle formation forming [12], finite-time tracking with bounded control input [13], event-triggered control [14], robust consensus control [15] and cooperative optimal control for multi-agent systems [16] have been studied, to name just a few. However, the dynamics of a large class of mechanical systems, including autonomous vehicles, walking robots and robotic manipulators are much more complicated than those linear models in practice [17–21]. Thus, it is significant and non-trivial to study the distributed coordination problem for multiple Lagrangian systems which can represent the motion of a great many mechanical systems.

Recently, some research works have been conducted to investigate the distributed cooperative control of the networked Lagrangian systems. One common but important problem is the distributed leaderless consensus problem that has been considered in [22–24]. Chopra and Spong [22] provided a passivity-based control framework for the coordination and synchronisation of multiple Lagrangian systems. In [23], a unified architecture of the consensus problem for networked Lagrangian systems based on passivity property was established. It should be noted that the methods used in [22, 23] are limited to the condition that the topology is balanced. The topology requirements were further relaxed in [24], where the leaderless synchronisation problem was studied under a directed graph containing a directed spanning tree. The above mentioned references are all aiming at driving the states of each system to unprescribed values decided by the corresponding initial states.

Another critical issue on distributed control polices for multiple Lagrangian systems is the leader-following problem including the containment control with multiple leaders and the coordinated tracking with one single leader, where the followers are expected to converge to the convex hull formed by the multiple leaders or track the desired reference trajectory, respectively. The containment control problem for Lagrangian systems were concerned in [25, 26]. The convergence of the followers' states to the dynamic convex hull was achieved with the aid of a distributed sliding-mode estimator and a non-singular sliding surface in [25]. Whereas, the results in [26] are only suitable in the case that the graph associated with the followers is undirected. In [26], containment control problem was realised under directed graph which is more general than that in [25]. The coordinated tracking to a reference trajectory for networked Lagrangian systems has been addressed in [24, 27–30]. Nuno *et al.* [24]

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considered the trajectory tracking of multiple Lagrange systems with unknown parameters coupled through a communication network with transmission delays. Contraction theory was introduced to cope with the tracking problem in [27] under an undirected ring graph. Nevertheless, the results in the above two references are only valid in the case that the leader is available to all the followers, which is rather restrictive in reality. Under the assumption that the interaction topology is undirected, a distributed adaptive controller in [28] was designed to track the reference trajectory for multiple mechanical systems. The work in [29] extended the topology requirements in [28] to directed graph containing a spanning tree by using neural network. However, the position and velocity tracking errors for each vehicle can only be guaranteed to be cooperatively uniformly ultimately bounded (UUB). Mei *et al.* [30] presented a class of model-independent sliding mode control law for networked Lagrangian systems with a dynamic leader such that the tracking errors converge to zero asymptotically at the cost of utilising two-hop communication information. However, it should be noted that the full state measurements are required to implement all the above proposed control algorithms.

Nevertheless, in some robotic applications, it may not always be possible to measure joint velocities because of unavailability of velocity sensors. Moreover, the velocity measurements delivered by tachometers are often contaminated by noise, which would reduce the dynamic performance of the system. Hence, the velocity information is often derived via numerical differentiation of position measurements. However, it is worthy to note that, under realistic working conditions, the observer-controller scheme would perform much better than the numerical differentiation method. Therefore it is of great significance to study the observer-based distributed control problem. Distributed control for double integrators using only position measurements are presented in [31, 32]. It is shown in [31] that, using the super-twisting algorithms, the containment control problem can be solved under an undirected graph. In [32], based on the construction of Henneberg sequence, an acyclic rigid formation can be achieved using relative position measurements, whereas the acceleration of the global leader is required to converge to zero when time goes to infinity. A linear observer was designed in [33] to deal with the leaderless consensus problem that the agents could not obtain the velocity information under an undirected topology. Then, the topology condition was extended to directed graph in [34, 35], where an adaptive gain and a velocity observer were introduced, respectively, to free the use of the velocity information. Nevertheless, the approach in [34] relies on the assumption that the leaders keep stationary and the absolute velocity information is available for all the agents. The tracking control using only position measurements is achieved in [35], which was extended by taking the model uncertainties and external disturbances into consideration in [36]. The sliding mode estimator and super-twisting algorithm are applied, respectively, to estimate the position and velocity of the leader in finite time in [36]. It is shown that both of the estimators are independent of the tracking error, which might cause chattering before the estimation errors converge to zero. In [37], under the undirected topology between followers, an observer-based controller was proposed based on the position measurements; however, the result is valid only when the leader keeps static. To cope with the tracking problem with a dynamic leader, the authors employed the 'two-step' method, which the same as that in [36]. The so-called 'two-step' is that, firstly, we obtain the position and velocity of the leader in finite time through the sliding-mode estimators or other techniques, then the controllers for tracking are designed on the basis of the estimations.

In particular, this paper studies the distributed synchronisation control problem of multiple Lagrangian systems under directed communication topologies. A new kind of observer is constructed to estimate the unavailable velocity information, in which the control gains can be freely chosen under some conditions. The contributions of this paper are twofold. Firstly, the novel distributed observer-based control scheme is proposed and stability analysis of the closed-loop system is well discussed. As a result, neither the relative velocity nor the absolute velocity is necessary to track the dynamic leader, which is more flexible than [34, 37]. Secondly, the feature of the proposed protocol is that the consensus errors are locally exponentially stable instead of uniformly ultimately bounded as that in [35, 36].

The rest of this paper is organised as follows. Section 2 introduces some basic properties about the Lagrangian system and the results on graph theory. In Section 3, the whole distributed tracking control framework consisting of the velocity observer and the distributed control strategy is introduced, followed by the stability analysis. In Section 4, the distributed observer-based controller for leaderless synchronisation problem and the stability proof are presented. Numerical examples are carried out to show the effectiveness of the proposed control algorithm in Section 5. Section 6 gives the conclusion.

2 Background

A group of n mechanical systems labelled as agents 1 to n are considered as followers. The dynamics of the ith agent is described by Euler–Lagrange equation as follows

$$M_i(q_i)\ddot{q}_i + C_i(q_i, \dot{q}_i)\dot{q}_i + G_i(q_i) = \tau_i, \quad i = 1, \dots, n \tag{1}$$

where $q_i \in \mathbb{R}^p$ is the vector of generalised coordinates, $M_i(q_i) \in \mathbb{R}^{p\times p}$ is the symmetric positive-definite inertia matrix, $C_i(q_i, \dot{q}_i)\dot{q}_i \in \mathbb{R}^p$ is the vector of coriolis and centrifugal torques, $G_i(q_i)$ is the vector of gravitational torques, and $\tau_i \in \mathbb{R}^p$ is the vector of control torque on agent i.

In the following, we use $M(q) \triangleq \text{diag}[M_1(q_1), \dots, M_n(q_n)]$, $C(q, \dot{q}) \triangleq \text{diag}[C_1(q_1, \dot{q}_1), \dots, C_n(q_n, \dot{q}_n)]$, $G(q) \triangleq [G_1^{\text{T}}(q_1), \dots, G_n^{\text{T}}(q_n)]^{\text{T}}$ as the vector form of $M_i(q_i)$, $C_i(q_i, \dot{q}_i), G_i(q_i)$, respectively.

For (1), the following properties hold [38, 39].

Property (1) For any i, there exist positive constants M_M and M_m, such that $0 < M_m I_p \le M_i(q_i) \le M_M I_p$.

Property (2) $\dot{M}_i(q_i) - 2C_i(q_i, \dot{q}_i)$ is skew symmetric, that is

$$\xi^{\text{T}}[\dot{M}_i(q_i) - 2C_i(q_i, \dot{q}_i)]\xi = 0, \quad \forall \xi \in \mathbb{R}^p \tag{2}$$

Property (3) For all $q_i, x, y, z \in \mathbb{R}^p$, $C_i(q_i, \dot{q}_i)$ in (1) satisfies

$$C_i(q_i, x)y = C_i(q_i, y)x \tag{3}$$

$$C_i(q_i, x + y)z = C_i(q_i, x)z + C_i(q_i, y)z \tag{4}$$

Remark 1: As we discussed before, Lagrangian systems modelled by (1) are often used to represent mechanical systems such as manipulators and walking robot. Thus, the states of the systems are normally bounded in real applications, and the values of M_M and M_m can be determined.

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Notations: The superscript T means transpose for real matrices. Let $\mathbf{1}_n$ and $\mathbf{0}_n$ denote, respectively, the $n \times 1$ column vectors with all entries equal to one and zero. I_n represents the identity matrix of dimension n. The Kronecker product of matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$ is defined as $A \otimes B$ [40]. $\|X\|$ denotes the Euclidean norm of matrix X.

We use $\mathcal{G}_n \triangleq (\mathcal{V}, \mathcal{E})$ to represent the interactions among the agents 1 to n with the node set $\mathcal{V} \triangleq \{1, \ldots, n\}$ and the edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. The edge (i, j) denotes that the agent i transmits information to agent j in a directed graph, but not vice versa. In an undirected graph, an edge $(i, j) \in \mathcal{E}$ if agent i and j can receive information from each other. Here, it is assumed that there is no loop in the graph, that is, $(i, i) \notin \mathcal{E}$. If an edge $(i, j) \in \mathcal{E}$, then we call node i is a neighbour of node j. Thus, the neighbour set of agent i is defined as $\mathcal{N}_i \triangleq \{j | (j, i) \in \mathcal{E}\}$. The root is a node that has directed paths to all the other nodes in a directed graph. A directed tree contains exactly one root and every other node has only one parent. A directed tree is called a directed spanning tree if it consists of all the nodes in a graph. A directed graph contains a directed spanning tree as long as one of its sub-graphs is a directed spanning tree. The adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$ is defined such that $a_{ij} > 0$ if $(j, i) \in \mathcal{E}$, and $a_{ij} = 0$ otherwise. Define the weighted in-degree of node i as $d_i = \sum_{j\in\mathcal{N}_i} a_{ij}$. Let the Laplacian matrix $\mathcal{L}_n = [l_{ij}] \in R^{n\times n}$, with $l_{ii} = \sum_{j=1, j\neq i}^{n} a_{ij}$ and $l_{ij} = -a_{ij}, i \neq j$.

Lemma 1 (Schur Complement [41]): Suppose Q and R are symmetric. The linear matrix inequality: $\begin{bmatrix} Q & S \\ S^{\mathrm{T}} & R \end{bmatrix} > 0$ is equivalent to $R > 0$ and $Q - SR^{-1}S^{\mathrm{T}} > 0$.

Lemma 2 [42]: If the directed graph $\mathcal{G}_n$ has a directed spanning tree, then there exist $\mathbf{1}_n$ satisfying $\mathcal{L}_n\mathbf{1}_n = \mathbf{0}_n$ and $\nu = [\nu_1, \ldots, \nu_n]^{\mathrm{T}}$ with $\nu_i \geq 0$ satisfying $\mathcal{L}_n^{\mathrm{T}}\nu = \mathbf{0}_n$ and $\mathbf{1}_n^{\mathrm{T}}\nu = 1$.

Lemma 3 [42]: Suppose that the adjacency matrix $\mathcal{A}$ is constant. The system $\dot{\xi} = -[\mathcal{L}_n(t) \otimes I_m]\xi$ achieves consensus asymptotically if and only if the directed graph $\mathcal{G}_n$ has a directed spanning tree. In particular, $\xi_i(t) \to \sum_{i=1}^{n} \nu_i\xi_i(0)$, as $t \to \infty$, where $\nu = [\nu_1, \ldots, \nu_n]^{\mathrm{T}} \geq 0, \mathbf{1}_n^{\mathrm{T}}\nu = 1$, and $\mathcal{L}_n^{\mathrm{T}}\nu = \mathbf{0}_n$.

3 Distributed tracking control with a dynamic leader

In this section, the distributed tracking control under directed graph is considered. Given a desired trajectory q_d (i.e. the dynamic leader)

$$\ddot{q}_d(t) = f(t, q_d(t), \dot{q}_d(t)) \tag{5}$$

where $f : \mathbb{R} \times \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^p$ is a uniformly continuously differentiable vector-valued function, q_d ($\dot{q}_d$ and $\ddot{q}_d$, respectively) is the position, (velocity and acceleration, respectively) of the leader. The desired trajectory q_d and its derivative are assumed to be available only to a portion of the followers. The objective is to propose a distributed control law τ_i for system i using only local position measurements q_i and $q_j, j \in \mathcal{N}_i$, such that

$$\lim_{t\to\infty}(q_i(t) - q_d(t)) = \mathbf{0}_p, \quad \lim_{t\to\infty}(\dot{q}_i(t) - \dot{q}_d(t)) = \mathbf{0}_p \tag{6}$$

holds. Furthermore, the desired trajectory $q_d(t)$ is assumed to satisfy the following assumption.

Assumption 1: The desired trajectory $q_d(t)$ is a bounded smooth signal with bounded derivative, that is, $\|q_d(t)\| \leq \bar{q}_d$, $\|\dot{q}_d(t)\| \leq \bar{\dot{q}}_d$, where $\bar{q}_d$ and $\bar{\dot{q}}_d$ are positive constants.

Definition 1 [35]: The auxiliary variables of position and velocity errors for systems i are defined as

$$s_i \triangleq q_i - \frac{1}{d_i + b_i}\left(\sum_{j\in\mathcal{N}_i} a_{ij}q_j + b_i q_d\right) \triangleq q_i - q_{si} \tag{7}$$

$$\dot{s}_i \triangleq \dot{q}_i - \frac{1}{d_i + b_i}\left(\sum_{j\in\mathcal{N}_i} a_{ij}\dot{q}_j + b_i\dot{q}_d\right) \triangleq \dot{q}_i - \dot{q}_{si} \tag{8}$$

where $b_i > 0$ if the leader is the neighbour of agent i and $b_i = 0$ otherwise. Accordingly, d_i is the weighted in-degree of agent i.

The vector form of the error dynamics can be written as

$$s = ((\bar{L} + B) \otimes I_p)(q - \mathbf{1}_n \otimes q_d) \tag{9}$$

$$\dot{s} = ((\bar{L} + B) \otimes I_p)(\dot{q} - \mathbf{1}_n \otimes \dot{q}_d) \tag{10}$$

where $s = [s_1^{\mathrm{T}}, s_2^{\mathrm{T}}, \ldots, s_n^{\mathrm{T}}]^{\mathrm{T}}$, $q = [q_1^{\mathrm{T}}, q_2^{\mathrm{T}}, \ldots, q_n^{\mathrm{T}}]^{\mathrm{T}}$ (see equation at the bottom of the page)

and $B = \operatorname{diag}\left\{\frac{b_1}{d_1+b_1}, \ldots, \frac{b_n}{d_n+b_n}\right\}$.

Lemma 4 [35, 43]: If the graph has a spanning tree with the leader as the root, then $s = 0$ and $\dot{s} = 0$ if and only if the tracking objective is achieved, that is, $q = \mathbf{1}_n \otimes q_d$ and $\dot{q} = \mathbf{1}_n \otimes \dot{q}_d$.

3.1 Observer-controller scheme

To deal with the challenging problem that only position measurements can be used, a velocity observer is designed to provide accurate velocity estimations. In this paper, $\hat{q}_i$ and $\hat{\dot{q}}_i$ are used to denote the estimates of the position q_i and velocity $\dot{q}_i$, respectively. Before moving on, the following auxiliary variables are needed.

$$\bar{L} = \begin{bmatrix} \dfrac{d_1}{d_1+b_1} & -\dfrac{a_{12}}{d_1+b_1} & \cdots & \cdots & -\dfrac{a_{1n}}{d_1+b_1} \\ -\dfrac{a_{21}}{d_2+b_2} & \dfrac{d_2}{d_2+b_2} & \cdots & \cdots & -\dfrac{a_{2n}}{d_2+b_2} \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ -\dfrac{a_{n1}}{d_n+b_n} & \cdots & \cdots & -\dfrac{a_{n(n-1)}}{d_n+b_n} & \dfrac{d_n}{d_n+b_n} \end{bmatrix}$$

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The position 'estimation' error is given as

$$\tilde{q}_i = \hat{q}_i - q_i \tag{11}$$

Accordingly, the velocity estimation error is

$$\dot{\tilde{q}}_i = \dot{\hat{q}}_i - \dot{q}_i \tag{12}$$

And, the combined estimation error is defined as

$$\eta_i = \dot{\tilde{q}}_i + \alpha\tilde{q}_i \tag{13}$$

where α is a positive constant to balance the velocity and position convergence rate.

To cope with the problem that only the position measurement of each follower is available, based on the dissipation theory, a velocity observer for ith system is specially designed as follows, in which only the local position information is used

$$\begin{aligned}\dot{\hat{q}}_i &= w_i - k_i\tilde{q}_i \\ \dot{w}_i &= M_i(q_i)^{-1}[\tau_i - C_i(q_i, \dot{\hat{q}}_i)(\dot{\hat{q}}_i + \alpha\tilde{q}_i) - G_i(q_i)] - l_i\tilde{q}_i\end{aligned} \tag{14}$$

where $k_i > \alpha$, $l_i = \alpha(k_i - \alpha)$ are user-chosen constant scalars, w_i is the intermediate variable.

Remark 2: Partially motivated by the idea that the observer is designed to make the storage function decrease in [44, 45], where the tracking problem for a single robot is considered. In [44, 45], the desired reference trajectory is always available to the certain follower. Whereas, in this paper, the distributed tracking problem of multiple Lagrangian systems is considered, in which not all the followers can obtain the reference trajectory.

Remark 3: To estimate the velocity information, the accurate position information is required in the observer. Generally, the position sensors, such as encoders, could give us very accurate measurements of the joint displacements. Even under the condition that the position measurements are contaminated by noise, we can introduce the robust term into the distributed observer to eliminate the influence caused by the noise.

Based on the estimated states, the distributed control law using only position measurements is proposed as

$$\tau_i = M_i(q_i)(\hat{\ddot{q}}_{si} + s_i) + C_i(q_i, \dot{\hat{q}}_i)(\hat{\dot{q}}_{si} - s_i) - \gamma_i(\hat{\dot{s}}_i + s_i) + G_i \tag{15}$$

where γ_i are positive constant, $\hat{\ddot{q}}_{si}, \hat{\dot{q}}_{si}$ and $\hat{\dot{s}}_i$ are the estimates of the variables $\ddot{q}_{si}$, $\dot{q}_{si}$ and $\dot{s}_i$, respectively. Accordingly, from (7) and (8), they have the following form

$$\begin{aligned}\hat{\dot{q}}_{si} &= \frac{1}{d_i + b_i}\left(\sum_{j\in\mathcal{N}_i} a_{ij}\dot{\hat{q}}_j + b_i\dot{q}_d\right) \\ \hat{\ddot{q}}_{si} &= \frac{1}{d_i + b_i}\left(\sum_{j\in\mathcal{N}_i} a_{ij}\ddot{\hat{q}}_j + b_i\ddot{q}_d\right) \\ \hat{\dot{s}}_i &= \dot{\hat{q}}_i - \hat{\dot{q}}_{si}\end{aligned} \tag{16}$$

3.2 Stability analysis

Theorem 1: Suppose that the communication topology among the $n+1$ agents contains a directed spanning tree with the leader as the root and no cycle exists. Under assumption 1, the tracking and velocity observation errors of multiple Lagrangian systems (1) locally exponentially converge to zero using the distributed observer (14) and control law (15), if parameters γ_i, α and k_i are selected such that

$$\min_i \gamma_i > M_M + \epsilon_1 + \epsilon_2 \tag{17}$$

$$\frac{\epsilon_1\bar{\sigma} - 1}{M_m\bar{\sigma} - \bar{\sigma} + \epsilon_1 + \epsilon_2} < \alpha \le \frac{-\alpha_b\bar{\sigma} + \sqrt{\alpha_b^2\bar{\sigma}^2 + 4\sqrt{n}\bar{\sigma}M_M}}{2\sqrt{n}\bar{\sigma}M_M} \tag{18}$$

$$0 < \min_i k_i \le \max_i k_i \le \frac{2\sqrt{n}M_M(P_{12}\phi_1 - \alpha\theta_2) - \phi_1^2 M_m + \sqrt{\Delta_3}}{2\alpha n M_M^2} \tag{19}$$

where ϵ_1 and ϵ_2 are positive parameters to be defined in (32). $\bar{\sigma}(\mathcal{L} + B)$ denotes the maximum singular value of matrix $\mathcal{L} + B$. ϕ_1 and P_{12} can be found in (35) and (37). The other variables α_b, θ_2 and Δ_3 are defined in Appendix 9.2.

Proof: The Lyapunov function candidate for the overall closed-loop system can be chosen as

$$V = V_0 + V_1 \tag{20}$$

with $V_0 = \frac{1}{2}\sum_{i=1}^n \eta_i^{\mathrm{T}} M_i(q_i)\eta_i + \frac{1}{2}\sum_{i=1}^n \tilde{q}_i^{\mathrm{T}}\tilde{q}_i$ and $V_1 = \frac{1}{2}\sum_{i=1}^n \xi_i^{\mathrm{T}} M_i(q_i)\xi_i$, where $\xi_i = \dot{s}_i + s_i$.

Taking the time derivative of V_0 along (12) and (13) gives

$$\begin{aligned}\dot{V}_0 &= \frac{1}{2}\sum_{i=1}^n \eta_i^{\mathrm{T}}\dot{M}_i(q_i)\eta_i + \sum_{i=1}^n \eta_i M_i(q_i)\dot{\eta}_i + \sum_{i=1}^n \tilde{q}_i^{\mathrm{T}}\dot{\tilde{q}}_i \\ &= \frac{1}{2}\sum_{i=1}^n \eta_i^{\mathrm{T}}\dot{M}(q_i)\eta_i + \sum_{i=1}^n \eta_i^{\mathrm{T}} M_i(q_i)(\ddot{\tilde{q}}_i + \alpha\dot{\tilde{q}}_i) \\ &\quad + \sum_{i=1}^n \tilde{q}_i^{\mathrm{T}}(\eta_i - \alpha\tilde{q}_i)\end{aligned} \tag{21}$$

In light of the observer (14), one has

$$M_i(q_i)\dot{w}_i = \tau_i - C_i(q_i, \dot{\hat{q}}_i)(\dot{\hat{q}}_i + \alpha\tilde{q}_i) - G_i(q_i) - l_i M_i\tilde{q}_i \tag{22}$$

and

$$M_i(q_i)\ddot{\hat{q}}_i = M_i(q_i)\dot{w}_i - k_i M_i(q_i)\dot{\tilde{q}}_i \tag{23}$$

Substituting equation (22) into (23), we can obtain

$$\begin{aligned}M_i(q_i)\ddot{\hat{q}}_i &= \tau_i - C_i(q_i, \dot{\hat{q}}_i)(\dot{\hat{q}}_i + \alpha\tilde{q}_i) - G_i(q_i) - l_i M_i\tilde{q}_i \\ &\quad - k_i M_i(q_i)\dot{\tilde{q}}_i\end{aligned} \tag{24}$$

Also, it can be easily obtained from the system model (1) that

$$M_i(q_i)\ddot{q}_i = \tau_i - C_i(q_i, \dot{q}_i)\dot{q}_i - G_i(q_i) \tag{25}$$

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Taking derivative on the both sides of velocity estimation error (12), and multiplying $M_i(q_i)$, we have

$$M_i(q_i)\ddot{\tilde{q}}_i = M_i(q_i)\ddot{\hat{q}}_i - M_i(q_i)\ddot{q}_i \quad (26)$$

Then, replacing the two terms on the right side of equation (26) with (24) and (25), one has

$$\begin{aligned} M_i(q_i)\ddot{\tilde{q}}_i = C_i(q_i,\dot{q}_i)\dot{q}_i - C_i(q_i,\dot{\hat{q}}_i)(\dot{\hat{q}}_i + \alpha\tilde{q}_i) \\ - k_i M_i \dot{\tilde{q}}_i - l_i M_i \tilde{q}_i \end{aligned} \quad (27)$$

Substituting equation (27) into (21), one has

$$\begin{aligned} \dot{V}_0 = \frac{1}{2}\sum_{i=1}^{n} \eta_i^{\mathrm{T}} \dot{M}_i \eta_i + \sum_{i=1}^{n} \eta_i^{\mathrm{T}} C_i(q_i,\dot{q}_i)\dot{q}_i \\ - \sum_{i=1}^{n} \eta_i^{\mathrm{T}} C_i(q_i,\dot{\hat{q}}_i)\dot{\hat{q}}_i - \alpha\sum_{i=1}^{n} \eta_i^{\mathrm{T}} C_i(q_i,\dot{\hat{q}}_i)\tilde{q}_i \\ - \sum_{i=1}^{n} k_i \eta_i M_i(q_i)\dot{\tilde{q}}_i - \sum_{i=1}^{n} l_i \eta_i^{\mathrm{T}} M_i \tilde{q}_i \\ + \alpha\sum_{i=1}^{n} \eta_i^{\mathrm{T}} M_i(q_i)\dot{\tilde{q}}_i - \alpha\sum_{i=1}^{n} \tilde{q}_i^{\mathrm{T}}\tilde{q}_i + \sum_{i=1}^{n} \tilde{q}_i^{\mathrm{T}}\eta_i \end{aligned} \quad (28)$$

Replacing $\dot{q}_i$ out of bracket in the second term with $\dot{q}_i = \dot{\hat{q}}_i + \alpha\tilde{q}_i - \eta_i$, we have

$$\begin{aligned} \dot{V}_0 = \frac{1}{2}\sum_{i=1}^{n} \eta_i^{\mathrm{T}} \dot{M}_i(q_i) \eta_i - \sum_{i=1}^{n} \eta_i^{\mathrm{T}} C_i(q_i,\dot{q}_i)\eta_i \\ + \sum_{i=1}^{n} \eta_i^{\mathrm{T}} C_i(q_i,\dot{q}_i)\dot{\hat{q}}_i + \alpha\sum_{i=1}^{n} \eta_i^{\mathrm{T}} C_i(q_i,\dot{q}_i)\tilde{q}_i \\ - \sum_{i=1}^{n} \eta_i^{\mathrm{T}} C_i(q_i,\dot{\hat{q}}_i)\dot{\hat{q}}_i - \alpha\sum_{i=1}^{n} \eta_i^{\mathrm{T}} C_i(q_i,\dot{\hat{q}}_i)\tilde{q}_i \\ - \sum_{i=1}^{n} (k_i - \alpha)\eta_i^{\mathrm{T}} M_i(q_i)\eta_i - \alpha\sum_{i=1}^{n} \tilde{q}_i^{\mathrm{T}}\tilde{q}_i + \sum_{i=1}^{n} \tilde{q}_i^{\mathrm{T}}\eta_i \end{aligned} \quad (29)$$

Noting that, in (29), the first two terms are cancelled by property (2). And, by directly using property (3), the following four terms can be simplified as

$$\begin{aligned} & C_i(q_i,\dot{q}_i)\dot{\hat{q}}_i + \alpha C_i(q_i,\dot{q}_i)\tilde{q}_i - C_i(q_i,\dot{\hat{q}}_i)\dot{\hat{q}}_i - \alpha C_i(q_i,\dot{\hat{q}}_i)\tilde{q}_i \\ & \quad = -C_i(q_i,\dot{\hat{q}}_i)(\eta_i - \alpha\tilde{q}_i) - \alpha C_i(q_i,\dot{\hat{q}}_i)\tilde{q}_i \\ & \quad = -C_i(q_i,\dot{\hat{q}}_i)\eta_i + \alpha C_i(q_i,\dot{\hat{q}}_i)\tilde{q}_i - \alpha C_i(q_i,\dot{\hat{q}}_i)\tilde{q}_i \end{aligned} \quad (30)$$

Defining the variable $x \triangleq \{\dot{\tilde{q}}_i, e_i, \dot{e}_i\}^{\mathrm{T}}$, where $e_i = q_i - q_d$, $\dot{e}_i = \dot{q}_i - \dot{q}_d$ and the set Ω_r

$$\Omega_r \triangleq \left\{x : |\frac{1}{2}\dot{\tilde{q}}_i^{\mathrm{T}}\dot{\tilde{q}}_i + \frac{1}{2}e_i^{\mathrm{T}}e_i + \frac{1}{2}\dot{e}_i^{\mathrm{T}}\dot{e}_i \le r, \quad \forall t \ge 0\right\}, \quad i = 1, \ldots, n \quad (31)$$

For any $x \in \Omega_r$, it can be deduced that $\dot{\hat{q}}_i$ and $\dot{q}_i$ are bounded. we note that for $C_i(q_i,\dot{\hat{q}}_i)\zeta$ and $C_i(q_i,\dot{q}_i)\zeta$ equal to zero when $\zeta = 0$. From the locally Lipschitz property of the coriolis and centrifugal torque, there exist positive constants ϵ_1 and ϵ_2 depending only upon r and $\dot{\bar{q}}_d$ such that

$$\|C_i(q_i,\dot{\hat{q}}_i)\zeta\| \le \epsilon_1\|\zeta\| \quad \text{and} \quad \|C_i(q_i,\dot{q}_i)\zeta\| \le \epsilon_2\|\zeta\| \quad (32)$$

By the properties of vector norm, we also have

$$\begin{aligned} \|C_i(q_i,\dot{\hat{q}}_i + \dot{q}_i)\zeta\| &= \|C_i(q_i,\dot{\hat{q}}_i)\zeta + C_i(q_i,\dot{q}_i)\zeta\| \\ &\le \|C_i(q_i,\dot{\hat{q}}_i)\zeta\| + \|C_i(q_i,\dot{q}_i)\zeta\| \\ &\le (\epsilon_1 + \epsilon_2)\|\zeta\| \\ \|C_i(q_i,\dot{\hat{q}}_i - \dot{q}_i)\zeta\| &= \|C_i(q_i,\dot{\hat{q}}_i)\zeta - C_i(q_i,\dot{q}_i)\zeta\| \\ &\le \|C_i(q_i,\dot{\hat{q}}_i)\zeta\| + \|C_i(q_i,\dot{q}_i)\zeta\| \\ &\le (\epsilon_1 + \epsilon_2)\|\zeta\| \end{aligned} \quad (33)$$

Combining (30) and (32), $\dot{V}_0$ can be upper bounded by

$$\begin{aligned} \dot{V}_0 \le -[M_m(\min_i k_i - \alpha) - \epsilon_1]\|\eta\|^2 - \alpha\|\tilde{q}\|^2 \\ + (1 + \alpha\epsilon_1 + \alpha\epsilon_2)\|\eta\|\|\tilde{q}\| \end{aligned} \quad (34)$$

Similar to the calculation procedure of $\dot{V}_0$, we can obtain (see (35))

Owing to the organisation of the paper, please refer to Appendix A for the details. Note that, the Lyapunov function candidate for the overall observer-controller scheme $V = V_0 + V_1$ satisfies

$$c_m \left\| \begin{bmatrix} \tilde{q} \\ \eta \\ \xi \end{bmatrix} \right\|^2 \le V(\tilde{q}, \eta, \xi) \le c_M \left\| \begin{bmatrix} \tilde{q} \\ \eta \\ \xi \end{bmatrix} \right\|^2 \quad (36)$$

for any positive constants c_m and c_M satisfying $c_m \le \min\{1, M_m\}$ and $c_M \ge \max\{1, M_M\}$

$$\begin{aligned} \dot{V}_1 \le & -[\min_i \gamma_i - M_M - (\epsilon_1 + \epsilon_2)]\|\xi\|^2 + \alpha\left[\frac{\sqrt{n}M_M}{M_m} + \sqrt{n}M_M\alpha + \epsilon_1\sqrt{n} + \epsilon_2 + \max_i \gamma_i \bar{\sigma}(\mathcal{L} + B)\right]\|\xi\|\|\tilde{q}\| \\ & + \left[\frac{\sqrt{n}M_M}{M_m}(\epsilon_1 + \epsilon_2) + \sqrt{n}M_M \max_i k_i + \epsilon_1\sqrt{n} + \epsilon_2 + \alpha\max_i \gamma_i \bar{\sigma}(\mathcal{L} + B)\right]\|\xi\|\|\eta\| \\ \triangleq & -[\min_i \gamma_i - M_M - (\epsilon_1 + \epsilon_2)]\|\xi\|^2 + \phi_1\|\xi\|\|\tilde{q}\| + \phi_2\|\xi\|\|\eta\| \end{aligned} \quad (35)$$

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Taking the time derivative of $V(y)$ along (34) and (35) yields

$$\dot{V}(y) = \dot{V}_0 + \dot{V}_1 \le -y^{\mathrm{T}} P y \tag{37}$$

where (see equation at the bottom of the page)

and $y = (\|\tilde{q}\|, \|\eta\|, \|\xi\|)^{\mathrm{T}}$.

Hence, we can derive from Lemma 1 that P is positive definite if (17)–(19) hold. Therefore it follows

$$\dot{V} \le -\lambda_{\min}(P) \left\| \begin{bmatrix} \tilde{q} \\ \eta \\ \xi \end{bmatrix} \right\|^2 \tag{38}$$

where $\lambda_{\min}(P)$ is the smallest eigenvalue of matrix P. Then, it follows from Theorem 4.10 in [46] that $[\tilde{q}, \eta, \xi]^{\mathrm{T}} = \mathbf{0}_{3p}$ is locally exponentially stable. Therefore it is guaranteed from $\xi = \dot{s} + s$ that s and $\dot{s}$ both converge to zero. By Lemma 4, $q_i - q_d \to \mathbf{0}_p$ and $\dot{q}_i - \dot{q}_d \to \mathbf{0}_p$, as $t \to \infty$, which implies that the tracking with a dynamic leader is achieved. One can also obtain that η and $\tilde{q}$ converge to zero from (36) and (38), which implies $q_i - \hat{q}_i \to \mathbf{0}_p$ and $\dot{q}_i - \dot{\hat{q}}_i \to \mathbf{0}_p$ as $t \to \infty$, $i = 1, \ldots, n$. Consequently, the velocity is estimated accurately through observer (14). Thus, it can be concluded that the proposed observer-controller scheme provides satisfactory distributed tracking control results for multiple Lagrangian systems without velocity measurements. □

Remark 4: It is worth mentioning that for any given r (i.e. the radius of the neighbourhood), ϵ_1 and ϵ_2 can be determined from (31), (32) by considering both r and the upper bound of the desired trajectory. Consequently, the bounds of the control gains in Theorem 1 can be obtained, which makes it very straightforward to set the feasible control gains only constrained by the controller capability.

Remark 5: In Theorem 1, the control parameters are chosen to satisfy the conditions (17)–(19). From (17), the lower bound of γ_i is independent of α and k_i, thus γ_i would not contradict them. Similarly, α is also not coupled with the other two parameters, and large ϵ_1 and ϵ_2 (hence, r) are expected to make α well defined, as well as provide more options to choose α. It can be founded that the lower bound of k_i can be determined if γ and α have been set previously. Hence, the three parameters can be selected without contradiction. According to Lemma 1, (17) can be directly derived from $\min_i \gamma_i - M_M - (\epsilon_1 + \epsilon_2) > 0$. Owing to the space limitation, please refer to Appendix 9.2 for the detailed calculations of (18) and (19).

4 Distributed leaderless synchronisation using only position measurements

In what follows, the distributed leaderless synchronisation problem for Lagrangian systems is discussed under a directed interaction topology. Our task is to design a distributed synchronisation algorithm such that all the agents achieve synchronisation, that is, the following equation holds

$$\lim_{t\to\infty} (q_i(t) - q_j(t)) = \mathbf{0}_p, \quad \lim_{t\to\infty} (\dot{q}_i(t) - \dot{q}_j(t)) = \mathbf{0}_p \tag{39}$$

In this section, the auxiliary sliding variable defined in (7) is changed as

$$s_i \triangleq \dot{q}_i - q_{ri} = \dot{q}_i + \beta \sum_{j\in\mathcal{N}_i} a_{ij}(q_i - q_j) \tag{40}$$

where β is a positive constant.

The vector form of s_i can be written as

$$s \triangleq \dot{q} - q_r = \dot{q} + \beta(\mathcal{L}_n \otimes I_p) q \tag{41}$$

where s, q_r and q are, respectively, the column vectors of s_i, q_{ri} and q_i.

The observer-based leaderless synchronisation control law is proposed for Lagrangian system (1) as

$$\tau_i = M_i(q_i)\hat{\dot{q}}_{ri} + C_i(q_i, \dot{\hat{q}}_i) q_{ri} + G_i(q_i) - k_p \hat{s}_i \tag{42}$$

where k_p is a positive control gain, $\hat{\dot{q}}_{ri}$ and $\hat{s}_i$ are the estimates of the variables $\dot{q}_{ri}$ and s_i, respectively. They are given as

$$\begin{aligned} \hat{\dot{q}}_{ri} &= -\beta \sum_{j\in\mathcal{N}_i} a_{ij}(\dot{\hat{q}}_i - \dot{\hat{q}}_j) \\ \hat{s}_i &= \dot{\hat{q}}_i - q_{ri} = \dot{\hat{q}}_i + \beta \sum_{j\in\mathcal{N}_i} a_{ij}(q_i - q_j) \end{aligned} \tag{43}$$

From (41) and (43), it can be derived that

$$\hat{s}_i = \dot{q}_i - q_{ri} - \dot{q}_i + \dot{\hat{q}}_i = s_i - \eta_i + \alpha\tilde{q}_i \tag{44}$$

Theorem 2: If the directed communication topology has a directed spanning tree, then the synchronisation errors locally exponentially converge to zero using the observer-contoller scheme (14) and (42), provided the parameters are chosen such that

$$\alpha > 0 \tag{45}$$

$$\min_i k_i > \frac{1}{4M_m\alpha}(1 + \alpha\epsilon_1 + \alpha\epsilon_2)^2 + \frac{\epsilon_1}{M_m} + \alpha \tag{46}$$

$$k_p > \frac{\alpha\phi_3^2(2 + \alpha\epsilon_1 + \alpha\epsilon_2) + \alpha^2\phi_3^2[M_m(\min_i k_i - \alpha) - \epsilon_1]}{4\alpha[M_m(\min_i k_i - \alpha) - \epsilon_1] - (1 + \alpha\epsilon_1 + \alpha\epsilon_2)^2} \tag{47}$$

where $\phi_3 = \beta M_M \overline{\sigma}(\mathcal{L}_n) + \epsilon_3$, and ϵ_3 is defined in (54). Also, ϵ_1 and ϵ_2 are the same as those in Theorem 1.

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} = \begin{bmatrix} \alpha & -\frac{1}{2}(1 + \alpha\epsilon_1 + \alpha\epsilon_2) & -\frac{1}{2}\phi_1 \\ -\frac{1}{2}(1 + \alpha\epsilon_1 + \alpha\epsilon_2) & M_m(\min_i k_i - \alpha) - \epsilon_1 & -\frac{1}{2}\phi_2 \\ -\frac{1}{2}\phi_1 & -\frac{1}{2}\phi_2 & \min_i \gamma_i - M_M - (\epsilon_1 + \epsilon_2) \end{bmatrix}$$

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Proof: From (40), substituting the sliding variable s_i into the system dynamic (1) yields

$$M_i\dot{s}_i + C_i(q_i,\dot{q}_i)s_i = \tau_i - M_i\ddot{q}_{ri} - C_i(q_i,\dot{q}_i)\dot{q}_{ri} - G_i \quad (48)$$

In terms of (43) and property (3), the following two equalities are presented here to facilitate the subsequent proof.

$$\begin{aligned}\dot{\hat{q}}_{ri} - \dot{q}_{ri} &= -\beta\sum_{j\in\mathcal{N}_i}(\dot{\hat{q}}_i - \dot{\hat{q}}_j) + \beta\sum_{j\in\mathcal{N}_i} a_{ij}(\dot{q}_i - \dot{q}_j)\\ &= \alpha\beta\sum_{j\in\mathcal{N}_i} a_{ij}(\tilde{q}_i - \tilde{q}_j) - \beta\sum_{j\in\mathcal{N}_i} a_{ij}(\eta_i - \eta_j) \quad (49)\end{aligned}$$

$$\begin{aligned}C_i(q_i,\dot{\hat{q}}_i)q_{ri} - C_i(q_i,\dot{q}_i)q_{ri} &= C_i(q_i,q_{ri})\dot{\tilde{q}}_i\\ &= C_i(q_i,q_{ri})\eta_i - \alpha C_i(q_i,q_{ri})\tilde{q}_i \quad (50)\end{aligned}$$

The sub-Lyapunov function candidate is chosen as

$$V_2 = \frac{1}{2}\sum_{i=1}^{n} s_i^{\mathrm{T}} M_i(q_i)s_i \quad (51)$$

The time derivative of V_2 along (48) is given as

$$\begin{aligned}\dot{V}_2 &= \frac{1}{2}\sum_{i=1}^{n} s_i^{\mathrm{T}}\dot{M}_i s_i + \sum_{i=1}^{n} s_i^{\mathrm{T}}\\ &\quad\times[-C_i(q_i,\dot{q}_i)s_i + \tau_i - M_i\dot{q}_{ri} - C_i(q_i,\dot{q}_i)q_{ri} - G_i(q_i)]\\ &= -k_p\sum_{i=1}^{n} s_i^{\mathrm{T}}s_i + \sum_{i=1}^{n} s_i^{\mathrm{T}}\\ &\quad\times\left[M_i(\dot{\hat{q}}_{ri} - \dot{q}_{ri}) + C_i(q_i,\dot{\hat{q}}_i)q_{ri} - C_i(q_i,\dot{q}_i)q_{ri}\right] \quad (52)\end{aligned}$$

Then, utilising property (2) and substituting (49) and (50) into (52) yields (see (53))

Assume that $x \in \Omega_r$, so q_i is bounded, yielding q_{ri} is also bounded from (40). Similar to (32), we can obtain

$$\|C_i(q_i,q_{ri})\zeta\| \le \epsilon_3\|\zeta\| \quad (54)$$

where ϵ_3 is a positive constant depending only upon r and $\bar{q}_d$. Therefore $\dot{V}_2$ can be upper bounded as

$$\dot{V}_2 \le -k_p\|s\|^2 + \alpha\phi_3\|s\|\|\tilde{q}\| + \phi_3\|s\|\|\eta\| \quad (55)$$

where $\phi_3 = \beta M_M\overline{\sigma}(\mathcal{L}_n) + \epsilon_3$

For the leaderless synchronisation case, the Lyapunov function candidate of the overall observer-controller scheme is chosen as

$$\begin{aligned}V = V_0 + V_2 &= \frac{1}{2}\sum_{i=1}^{n}\eta_i^{\mathrm{T}} M_i(q_i)\eta_i + \frac{1}{2}\sum_{i=1}^{n}\tilde{q}_i^{\mathrm{T}}\tilde{q}_i\\ &\quad + \frac{1}{2}\sum_{i=1}^{n} s_i^{\mathrm{T}} M_i(q_i)s_i \quad (56)\end{aligned}$$

satisfying

$$c_m\left\|\begin{bmatrix}\tilde{q}\\ \eta\\ s\end{bmatrix}\right\|^2 \le V(\tilde{q},\eta,s) \le c_M\left\|\begin{bmatrix}\tilde{q}\\ \eta\\ s\end{bmatrix}\right\|^2 \quad (57)$$

for any positive constants c_m and c_M satisfying $c_m \le \min\{1, M_m\}$ and $c_M \ge \max\{1, M_M\}$

The derivative of $V(z)$ can be upper bounded as follows

$$\dot{V}(z) = \dot{V}_0 + \dot{V}_2 \le -z^{\mathrm{T}}Qz \quad (58)$$

where

$$Q = \begin{bmatrix}\alpha & -\frac{1}{2}(1+\alpha\epsilon_1+\alpha\epsilon_2) & -\frac{\alpha}{2}\phi_3\\ -\frac{1}{2}(1+\alpha\epsilon_1+\alpha\epsilon_2) & M_m(\min_i k_i - \alpha) - \epsilon_1 & -\frac{1}{2}\phi_3\\ -\frac{\alpha}{2}\phi_3 & -\frac{1}{2}\phi_3 & k_p\end{bmatrix},$$

and $z = [\|\tilde{q}\|, \|\eta\|, \|s\|]^{\mathrm{T}}$. From the basic theorem that a matrix is positive definite if and only if all of its principal minor determinants are positive, we can obtain Q is positive definite if the conditions (45)–(47) are satisfied. Hence, we have

$$\dot{V} \le -\lambda_{\min}(Q)\left\|\begin{bmatrix}\tilde{q}\\ \eta\\ s\end{bmatrix}\right\|^2 \quad (59)$$

where $\lambda_{\min}(Q)$ is the smallest eigenvalue of matrix Q.

Therefore it is concluded from Theorem 4.10 in [46] that $[\tilde{q},\eta,s]^{\mathrm{T}} = \mathbf{0}_3$ is locally exponentially stable, thus s converges to zero, as $t\to\infty$. Then, considering the definition of sliding variable s in (41), we can conclude from Lemmas 1 and 2 that the systems can reach synchronisation with $q_i(t) \to \sum_{i=1}^{n}\nu_i q_i(0)$, where $\nu = [\nu_1,\ldots,\nu_n]^{\mathrm{T}}$, satisfying $\mathbf{1}_n^{\mathrm{T}}\nu = 1$, and $\mathcal{L}_n^{\mathrm{T}}\nu = \mathbf{0}_n$. Moreover, using the similar analysis as that in Theorem 1, it can be proved that the velocity observation errors $\dot{\tilde{q}}$ also converge to zero. □

Remark 6: It is worth pointing out that the control architecture in this paper can be applied to formation control of multi-agent systems. Suppose that the desired distance between agent i and agent j is d_{ij}. Then, the synchronisation error q_{ri} will be changed to $-\beta\sum_{j\in\mathcal{N}_i} a_{ij}(q_i - q_j - d_{ij})$. Under the condition that the communication graph contains a directed spanning tree, the formation control would be achieved, that is, $\lim_{t\to\infty}\|q_i(t) - q_j(t) - d_{ij}\| = 0$, $\lim_{t\to\infty}\dot{q}_i = 0, \forall i,j = 1,\ldots,n$.

$$\dot{V}_2 = -k_p\sum_{i=1}^{n} s_i^{\mathrm{T}}s_i + \sum_{i=1}^{n} s_i^{\mathrm{T}}\left\{\beta M_i\left[\alpha\sum_{j\in\mathcal{N}_i} a_{ij}(\tilde{q}_i - \tilde{q}_j) - \sum_{j\in\mathcal{N}_i} a_{ij}(\eta_i - \eta_j)\right] + C_i(q_i,q_{ri})\eta_i - \alpha C_i(q_i,q_{ri})\tilde{q}_i\right\} \quad (53)$$

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5 Simulation results

Numerical simulations are presented in this section to demonstrate the effectiveness of the proposed observer-controller scheme. For simplicity, we choose four identical networked two-link manipulators modelled by Euler–Lagrange equation. The readers can refer to [38] for details. Let the masses of links 1 and 2 be, respectively, $m_1 = 0.5\,\text{kg}$, and $m_2 = 0.4\,\text{kg}$, the lengths of links 1 and 2 be, respectively, $l_1 = 0.4\,\text{m}$ and $l_2 = 0.3\,\text{m}$, the distances of the mass centre of links 1 and 2 between neighbours be, respectively, $l_{c1} = 0.2\,\text{m}$, and $l_{c2} = 0.15\,\text{m}$. In addition, the moments of inertia of link 1 and link 2 are, respectively, $J_1 = 0.0067\,\text{kg}\,\text{m}^2$ and $J_2 = 0.003\,\text{kg}\,\text{m}^2$.

Case 1. Distributed tracking control with a dynamic leader

To facilitate the comparison with [35], the same interaction topology is chosen as shown in Fig. 1, in which the leader is denoted as agent 0. The initial position of the followers are chosen as $q_i(0) = [(\pi/5)i, (\pi/4)i]^{\mathrm{T}}$ rad, and the initial velocity observations of the four followers are selected as $\hat{\dot{q}}(0) = [0.1, 0.1]^{\mathrm{T}}$ rad/s. Let the reference states of the leader be $q_d(t) = [\sin(t), \cos(t)]^{\mathrm{T}}$ rad, and hence the angular velocity be $\dot{q}_d(t) = [\cos(t), -\sin(t)]^{\mathrm{T}}$ rad/s. From the conditions (17)–(19), the control parameters are set as $\gamma_i = 4$, $\alpha = 2$, and $k_i = 8$, $i = 1, \ldots, 4$.

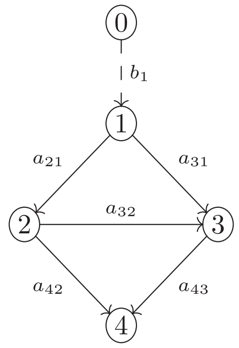

Fig. 1 *Network topology associated with the leader 0 and four followers. Here, an arrow from i to j denotes agent j can receive information form agent i, $i,j = 0, 1, \ldots, 4$.*

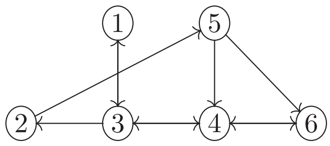

Fig. 3 *Directed communication graph of the leaderless synchronisation control system, where six agents are interacted in the topology.*

Fig. 2 illustrates the tracking and observation errors during the tracking process. From the first two pictures, it is obvious that the distributed tracking to a dynamic leader is achieved effectively. The performance of the observer is presented in Fig. 2c, which implies that the accurate estimations of the angular velocity information can be obtained. All the simulation results demonstrate the effectiveness and feasibility of the proposed framework of the observer-based controller.

In comparison with [35], in which the tracking errors are proved to be UUB, the errors under the proposed observer-controller can converge to zero with lower control gains under the same topology.

Case 2. Distributed leaderless synchronisation under a directed graph

Consider a group of six identical networked two-link revolute joint arms with the same initial states as that in Case 1. The same communication graph as [34] is given in Fig. 3, in which a directed spanning tree is contained. The control parameters are chosen as $\alpha = 2$, $k_i = 6$, $\beta = 2$ and $k_p = 3, i = 1, \ldots, 6$. The angle and angular velocity evolution during the synchronisation process are shown in Fig. 4. Compared with the results in [34], in which the absolute velocity information is very necessary to reach synchronisation, the proposed observer-controller scheme can achieve the leaderless synchronisation without using any velocity information even when the initial errors are obviously larger.

In order to further validate the effectiveness of the proposed observer-controller scheme for the systems with large size, the simulations of 30 agents under a directed communication topology have also been studied to verify the proposed control scheme. It can be concluded from Fig. 5 that because of the distributed characteristic, the proposed

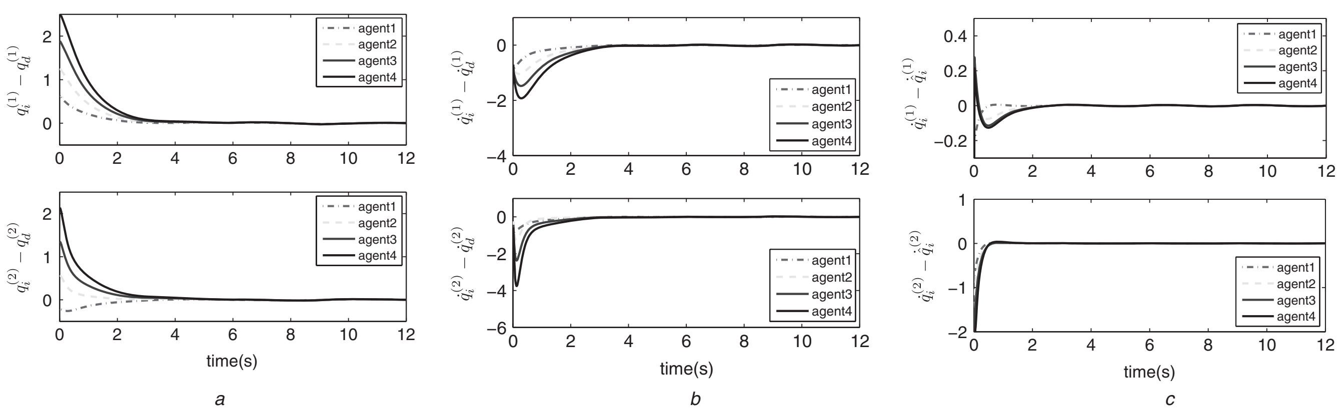

Fig. 2 *Tracking and observation errors during the evolution*

a Angle tracking errors
b Angular velocity tracking errors
c Velocity observation errors

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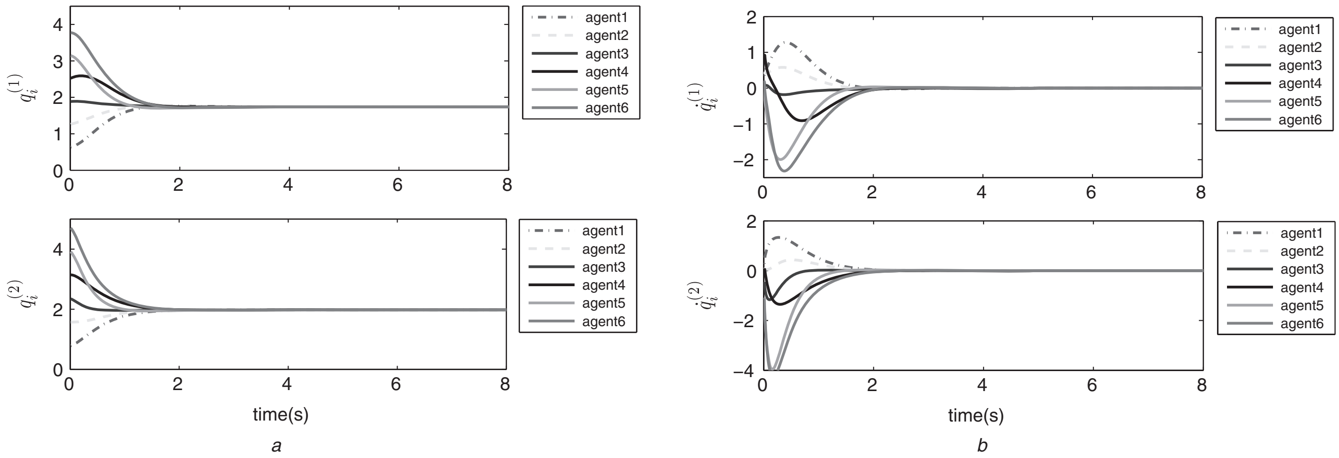

Fig. 4 *States variation under control law (42)*

a Angle variations

b Angular velocity variations

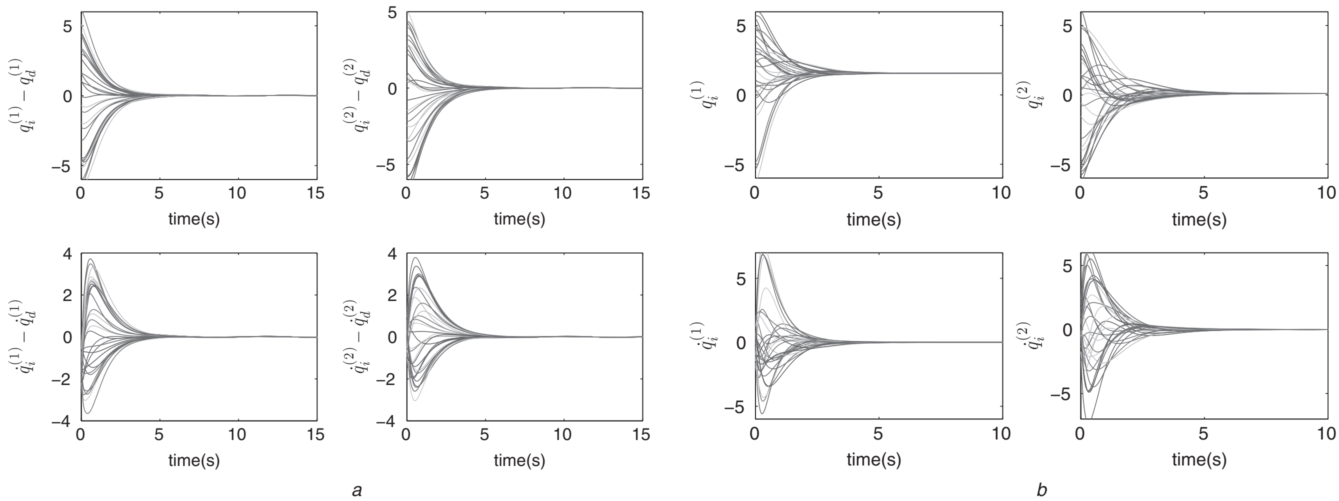

Fig. 5 *Response of the large size system to the proposed control scheme*

a Tracking errors

b States evolution of the leaderless case

observer-controller can also yield satisfactory control results for the large size of multiple Lagrangian systems based on the accurate velocity observation.

6 Conclusion

In this paper, the distributed coordination problem for multiple Lagrangian systems using only position measurements has been studied under a directed graph. A novel observer is elaborately designed based on the dissipation theory for each follower by using only local information to deal with the unavailability of velocity information. The estimated states are then used by the individual agents, respectively, to cooperatively track the dynamic leader and to realise leaderless synchronisation under the distributed observer-based control framework. The proposed observer-controller scheme can be extended to more general non-linear systems. Stability of the overall system has been proved and the effectiveness of the proposed distributed coordinated control laws is verified by simulation results. Future work will be focused on how to improve the scalability of the approach under the switching communication topology with time-delays.

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9 Appendix

9.1 Appendix A: The calculation of (35)

Before moving on, some preparations are made to facilitate the calculation of $\dot{V}_1(t)$. On one hand,

$$M_i\ddot{\hat{q}}_{si} - M_i\ddot{q}_{si} = \frac{M_i}{d_i + b_i}\left(\sum_{j\in\mathcal{N}_i} a_{ij}\ddot{\hat{q}}_j + b_i\ddot{q}_d - \sum_{j\in\mathcal{N}_i} a_{ij}\ddot{q}_j - b_i\ddot{q}_d\right) = \frac{M_i}{d_i + b_i}\sum_{j\in\mathcal{N}_i} a_{ij}(\ddot{\hat{q}}_j - \ddot{q}_j) \tag{60}$$

From (16) and the system model (1), it follows that (see equation (61) at the bottom of next page)

where, by property (3), the following transformation is utilised

$$\begin{aligned} &C_j(q_j,\dot{q}_j)\dot{q}_j - C_j(q_j,\dot{\hat{q}}_j)(\dot{\hat{q}}_j + \alpha\tilde{q}_j) \\ &= C_j(q_j,\dot{q}_j)\dot{q}_j - C_j(q_j,\dot{q}_j)\dot{\hat{q}}_j + C_j(q_j,\dot{q}_j)\dot{\hat{q}}_j \\ &\quad - C_j(q_j,\dot{\hat{q}}_j)(\dot{\hat{q}}_j + \alpha\tilde{q}_j) \\ &= C_j(q_j,\dot{q}_j)(\alpha\tilde{q}_j - \eta_j) + C_j(q_j,\dot{\hat{q}}_j)(\alpha\tilde{q}_j - \eta_j) \\ &\quad - C_j(q_j,\dot{\hat{q}}_j)\alpha\tilde{q}_j \\ &= C_j(q_j,\dot{q}_j)\alpha\tilde{q}_j - C_j(q_j,\dot{q}_j + \dot{\hat{q}}_j)\eta_j \end{aligned} \tag{62}$$

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On the other hand,

$$\begin{aligned}
&C_i(q_i,\hat{\dot{q}}_i)(\hat{\dot{q}}_{si}-s_i)-C_i(q_i,\dot{q}_i)(\dot{q}_{si}-s_i)\\
&=C_i(q_i,\hat{\dot{q}}_i)\hat{\dot{q}}_{si}-C_i(q_i,\hat{\dot{q}}_i)\dot{q}_{si}+C_i(q_i,\hat{\dot{q}}_i)\dot{q}_{si}\\
&\quad -C_i(q_i,\dot{q}_i)\dot{q}_{si}-C_i(q_i,\hat{\dot{q}}_i)s_i+C_i(q_i,\dot{q}_i)s_i\\
&=\frac{C_i(q_i,\hat{\dot{q}}_i)}{d_i+b_i}\sum_{j\in\mathcal{N}_i}a_{ij}(\hat{\dot{q}}_j-\dot{q}_j)\\
&\quad -C_i(q_i,\tilde{\dot{q}}_i)(\dot{s}_i-\dot{q}_i)-C_i(q_i,\tilde{\dot{q}}_i)s_i\\
&=\frac{C_i(q_i,\hat{\dot{q}}_i)}{d_i+b_i}\sum_{j\in\mathcal{N}_i}a_{ij}(\eta_j-\alpha\tilde{q}_j)\\
&\quad -C_i(q_i,\tilde{\dot{q}}_i)(\dot{s}_i+s_i)+C_i(q_i,\dot{q}_i)(\eta_i-\alpha\tilde{q}_i)
\end{aligned}\tag{63}$$

It can be derived from (25) and (40) that

$$\begin{aligned}
M_i(q_i)\ddot{s}_i&=M_i(q_i)(\ddot{q}_i-\ddot{q}_{si})\\
&=\tau_i-C_i(q_i,\dot{q}_i)\dot{q}_i-G_i(q_i)-M_i(q_i)\ddot{q}_{si}\\
&=\tau_i-C_i(q_i,\dot{q}_i)\dot{s}_i\\
&\quad -C_i(q_i,\dot{q}_i)\dot{q}_{si}-G_i(q_i)-M_i(q_i)\ddot{q}_{si}
\end{aligned}\tag{64}$$

Hence, the time derivative of V_1 is

$$\begin{aligned}
\dot{V}_1&=\frac{1}{2}\sum_{i=1}^{n}\xi_i^{\mathrm{T}}\dot{M}_i\xi_i+\sum_{i=1}^{n}\xi_i^{\mathrm{T}}M_i\dot{\xi}_i\\
&=\frac{1}{2}\sum_{i=1}^{n}\xi_i^{\mathrm{T}}\dot{M}_i\xi_i-\sum_{i=1}^{n}\xi_i^{\mathrm{T}}C_i(q_i,\dot{q}_i)\xi_i\\
&\quad+\sum_{i=1}^{n}\xi_i^{\mathrm{T}}[\tau_i-C_i(q_i,\dot{q}_i)(\dot{q}_{si}-s_i)\\
&\quad-M_i\ddot{q}_{si}-G_i(q_i)+M_i\dot{s}_i]
\end{aligned}\tag{65}$$

Then, substituting the control law (15) into the above equation, we obtain (see (66))

where we have used the fact that

$$\begin{aligned}
\hat{\dot{s}}_i&=\hat{\dot{q}}_i-\hat{\dot{q}}_{si}=\dot{q}_i-\dot{q}_{si}+(\hat{\dot{q}}_i-\dot{q}_i)+(\dot{q}_{si}-\hat{\dot{q}}_{si})\\
&=\dot{s}_i+\eta_i-\alpha\tilde{q}_i-\frac{1}{d_i+b_i}\sum_{j\in\mathcal{N}_i}a_{ij}(\eta_j-\alpha\tilde{q}_j)
\end{aligned}\tag{67}$$

$$\begin{aligned}
\frac{M_i}{d_i+b_i}\sum_{j\in\mathcal{N}_i}a_{ij}(\hat{\ddot{q}}_j-\ddot{q}_j)&=\frac{M_i}{d_i+b_i}\sum_{j\in\mathcal{N}_i}a_{ij}(\dot{w}_j-k_j\tilde{\dot{q}}_j-\ddot{q}_j)\\
&=\frac{M_i}{d_i+b_i}\sum_{j\in\mathcal{N}_i}a_{ij}\left\{M_j^{-1}\left[\tau_j-C_j(q_j,\hat{\dot{q}}_j)(\hat{\dot{q}}_j+\alpha\tilde{q}_j)-G_j(q_j)\right]\right.\\
&\quad\left.-l_j\tilde{q}_j-k_j\tilde{\dot{q}}_j-M_j^{-1}\left[\tau_j-C_j(q_j,\dot{q}_j)\dot{q}_j-G_j(q_j)\right]\right\}\\
&=\frac{M_i}{d_i+b_i}\sum_{j\in\mathcal{N}_i}a_{ij}\left\{M_j^{-1}\left[C_j(q_j,\dot{q}_j)\dot{q}_j-C_j(q_j,\hat{\dot{q}}_j)(\hat{\dot{q}}_j+\alpha\tilde{q}_j)\right]-k_j\tilde{\dot{q}}_j-l_j\tilde{q}_j\right\}\\
&=\frac{M_i}{d_i+b_i}\sum_{j\in\mathcal{N}_i}a_{ij}\left\{M_j^{-1}\left[\alpha C_j(q_j,\dot{q}_j)\tilde{q}_j-C_j(q_j,\dot{q}_j+\hat{\dot{q}}_j)\eta_j\right]-k_j\eta_j+\alpha^2\tilde{q}_j\right\}
\end{aligned}\tag{61}$$

$$\begin{aligned}
\dot{V}_1&=\sum_{i=1}^{n}\xi_i^{\mathrm{T}}\left[M_i\hat{\ddot{q}}_{si}-M_i\ddot{q}_{si}+C_i(q_i,\hat{\dot{q}}_i)(\hat{\dot{q}}_{si}-s_i)-C_i(q_i,\dot{q}_i)(\dot{q}_{si}-s_i)\right]\\
&\quad+\sum_{i=1}^{n}\xi_i^{\mathrm{T}}[-(\gamma_i-M_i)\xi_i]-\sum_{i=1}^{n}\xi_i^{\mathrm{T}}\gamma_i\left[\eta_i-\alpha\tilde{q}_i-\frac{1}{d_i+b_i}\sum_{j\in\mathcal{N}_i}a_{ij}(\eta_j-\alpha\tilde{q}_j)\right]\\
&=\sum_{i=1}^{n}\xi_i^{\mathrm{T}}\frac{M_i}{d_i+b_i}\sum_{j\in\mathcal{N}_i}a_{ij}\left\{M_j^{-1}\left[\alpha C_j(q_j,\dot{q}_j)\tilde{q}_j-C_j(q_j,\dot{q}_j+\hat{\dot{q}}_j)\eta_j\right]-k_j\eta_j+\alpha^2\tilde{q}_j\right\}\\
&\quad+\sum_{i=1}^{n}\xi_i^{\mathrm{T}}\left\{\frac{C_i(q_i,\hat{\dot{q}}_i)}{d_i+b_i}\sum_{j\in\mathcal{N}_i}a_{ij}(\eta_j-\alpha\tilde{q}_j)-C_i(q_i,\tilde{\dot{q}}_i)(\dot{s}_i+s_i)+C_i(q_i,\dot{q}_i)(\eta_i-\alpha\tilde{q}_i)\right\}\\
&\quad-\sum_{i=1}^{n}\xi_i^{\mathrm{T}}(\gamma_i-M_i)\xi_i-\sum_{i=1}^{n}\xi_i^{\mathrm{T}}\gamma_i\left[\eta_i-\alpha\tilde{q}_i-\frac{1}{d_i+b_i}\sum_{j\in\mathcal{N}_i}a_{ij}(\eta_j-\alpha\tilde{q}_j)\right]
\end{aligned}\tag{66}$$

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Recalling (32) and (54), for $x \in \Omega_r$, we have (see (68))

$$\dot{V}_1 \le -\left[\min_i \gamma_i - M_M - (\epsilon_1+\epsilon_2)\right]\|\xi\|^2 + \frac{\sqrt{n}M_M}{M_m}\|\xi\|(\alpha\epsilon_1\|\tilde{q}\| + (\epsilon_1+\epsilon_2)\|\eta\|)$$
$$+ \sqrt{n}M_M\|\xi\|(\max_i k_i\|\eta\| + \alpha^2\|\tilde{q}\|) + \|\xi\|\left[\epsilon_1\sqrt{n}(\|\eta\| + \alpha\|\tilde{q}\|) + \epsilon_2(\|\eta\| + \alpha\|\tilde{q}\|)\right]$$
$$+ \max_i \gamma_i \bar{\sigma}(\mathcal{L}+B)\|\xi\|(\alpha\|\tilde{q}\| + \|\eta\|)$$
$$= -\left[\min_i \gamma_i - M_M - (\epsilon_1+\epsilon_2)\right]\|\xi\|^2 + \alpha\left[\frac{\sqrt{n}M_M}{M_m} + \sqrt{n}M_M\alpha + \epsilon_1\sqrt{n} + \epsilon_2 + \max_i \gamma_i\bar{\sigma}(\bar{\mathcal{L}}+B)\right]\|\xi\|\|\tilde{q}\|$$
$$+ \left[\frac{\sqrt{n}M_M}{M_m}(\epsilon_1+\epsilon_2) + \sqrt{n}M_M\max_i k_i + \epsilon_1\sqrt{n} + \epsilon_2 + \alpha\max_i \gamma_i\bar{\sigma}(\bar{\mathcal{L}}+B)\right]\|\xi\|\|\eta\| \quad (68)$$

where $\bar{\sigma}(X)$ denotes the maximum singular value of the matrix X.

9.2 Appendix B: Calculations of condition (18) and (19)

In this appendix, the details on how to derive inequalities (18) and (19) will be presented.

For simplicity, let $\phi_1 \triangleq \alpha(\theta_1 + \bar{\sigma}\max_i \gamma_i)$ and $\phi_2 \triangleq \theta_2 + \sqrt{n}M_M\max_i k_i$, where $\theta_1 \triangleq \frac{\sqrt{n}M_M}{M_m} + \sqrt{n}M_M\alpha + \epsilon_1\sqrt{n} + \epsilon_2$ and $\theta_2 \triangleq \frac{\sqrt{n}M_M}{M_m}(\epsilon_1+\epsilon_2) + \epsilon_1\sqrt{n} + \epsilon_2 + \alpha\max_i \gamma_i\bar{\sigma}$, and $\bar{\sigma}$ represents for $\bar{\sigma}(\mathcal{L}+B)$. In this section, to coincide with Lemma 1, the matrix P is partitioned as

$$P = \left[\begin{array}{cc|c} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ \hline P_{31} & P_{32} & P_{33} \end{array}\right]$$

From Lemma 1, besides (17), which guarantees $P_{33} > 0$, the following condition also needs to be satisfied to make P positive definite

$$P_{b1} \triangleq \begin{bmatrix} \alpha - \dfrac{\phi_1^2}{4P_{33}} & P_{12} - \dfrac{\phi_1\phi_2}{4P_{33}} \\ P_{12} - \dfrac{\phi_1\phi_2}{4P_{33}} & M_m(\min_i k_i - \alpha) - \epsilon_1 - \dfrac{\phi_2^2}{4P_{33}} \end{bmatrix} > 0 \quad (69)$$

where P_{12} and P_{33} are components of matrix P, which are defined in (37). To make $P_{b1} > 0$, we have

$$\alpha - \frac{\phi_1^2}{4P_{33}} > 0 \quad (70)$$

$$|P_{b1}| > 0 \quad (71)$$

where $|P_{b1}|$ denotes the determinant of matrix P_{b1}.

Firstly, Considering (70), it is equivalent to

$$\alpha - \frac{\alpha^2}{4P_{33}}[\theta_1^2 + \bar{\sigma}^2(\max_i \gamma_i)^2 + 2\theta_1\bar{\sigma}\max_i \gamma_i] > 0 \quad (72)$$

Note that the inequality (72) contain parameters α and γ, then the condition for α can be deduced based on γ. Rewrite (72) as

$$\alpha\bar{\sigma}^2(\max_i \gamma_i)^2 + 2\alpha\bar{\sigma}\theta_1(\max_i \gamma_i) + \alpha\theta_1^2 - 4P_{33} < 0 \quad (73)$$

Here, we regard (73) as a second-order inequality with respect to $\max_i \gamma_i$. Considering that the quadratic coefficient $\alpha\bar{\sigma}^2$ is positive, and $\Delta_1 = 4\alpha^2\bar{\sigma}^2\theta_1^2 - 4\alpha\bar{\sigma}^2(\alpha\theta_1^2 - 4P_{33}) = 16\alpha\bar{\sigma}^2P_{33} > 0$, thus (73) always has solutions, namely, there always exists $\max_i \gamma_i$ satisfying (73). Nevertheless, recalling $P_{33} = \min_i \gamma_i - M_M - (\epsilon_1+\epsilon_2)$, it should be noted that both $\max_i \gamma_i$ and $\min_i \gamma_i$ are contained in the inequality, thus it might lead contradiction, that is, $\max_i \gamma_i < \min_i \gamma_i$. Hence, to avoid this situation, the following inequality should be satisfied

$$\frac{-2\alpha\bar{\sigma}\theta_1 + 4\bar{\sigma}\sqrt{\alpha[\min_i \gamma_i - (M_M + \epsilon_1 + \epsilon_2)]}}{2\alpha\bar{\sigma}^2} \ge \min_i \gamma_i \quad (74)$$

It can be reduced to

$$\alpha\bar{\sigma}^2(\min_i \gamma_i)^2 + (2\alpha\bar{\sigma}\theta_1 - 4)\min_i \gamma_i + \alpha\theta_1^2 + 4(M_M + \epsilon_1 + \epsilon_2) \le 0 \quad (75)$$

On one hand, (75) has solutions only when the following inequality holds

$$\Delta_2 = (2\alpha\bar{\sigma}\theta_1 - 4)^2 - 4\alpha\bar{\sigma}^2[\alpha\theta_1^2 + 4(M_M + \epsilon_1 + \epsilon_2)] \ge 0 \quad (76)$$

From (76), it yields

$$1 - \alpha\bar{\sigma}\theta_1 - \alpha\bar{\sigma}^2(M_M + \epsilon_1 + \epsilon_2) \ge 0 \quad (77)$$

In view of the definition of θ_1, then the condition of α is given by

$$0 < \alpha \le \frac{-\alpha_b\bar{\sigma} + \sqrt{\alpha_b^2\bar{\sigma}^2 + 4\sqrt{n}\bar{\sigma}M_M}}{2\sqrt{n}\bar{\sigma}M_M} \quad (78)$$

where $\alpha_b = \frac{\sqrt{n}M_M}{M_m} + \epsilon_1\sqrt{n} + \epsilon_2 + \bar{\sigma}(M_M + \epsilon_1 + \epsilon_2)$. It is easily verified that (78) is well defined under the condition (18). On the other hand, under the condition (17), the solution of (75) also needs to satisfy (see (79))

$$\frac{4 - 2\alpha\bar{\sigma}\theta_1 - \sqrt{(2\alpha\bar{\sigma}\theta_1 - 4)^2 - 4\alpha\bar{\sigma}^2[\alpha\theta_1^2 + 4(M_M + \epsilon_1 + \epsilon_2)]}}{2\alpha\bar{\sigma}^2} > M_M + \epsilon_1 + \epsilon_2 \quad (79)$$

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After algebraic operations, it turns to be

$$\alpha\bar{\sigma}^2(M_M + \epsilon_1 + \epsilon_2)^2 + 2\alpha\bar{\sigma}\theta_1(M_M + \epsilon_1 + \epsilon_2) + \alpha\theta_1^2 > 0 \quad (80)$$

It can be seen that (80) obviously holds in any condition, since all the parameters are positive, which in turn implies (18) makes sense for (70).

Then, consider (71), from which the condition for k_i can be deduced based on (17) and (18). Firstly, by the definition of P_{b1} in (69), computing (71) and separating the parameter k_i with others, we have

$$\begin{aligned}&\alpha n M_M^2(\max_i k_i)^2 + 2(\alpha\theta_2\sqrt{n}M_M - P_{12}\phi_1\sqrt{n}M_M)\max_i k_i\\&+ \phi_1^2 M_m \min_i k_i + c < 0\end{aligned} \quad (81)$$

where $c = \alpha\theta_2^2 - 2P_{12}\phi_1\theta_2 + 4P_{12}^2P_{33} - (\phi_1^2 - 4\alpha P_{33})\ (M_m \alpha - \epsilon_1)$. Next, replacing $\min_i k_i$ with $\max_i k_i$, and computing the discriminant

$$\Delta_3 = (2\alpha\theta_2\sqrt{n}M_M - 2P_{12}\phi_1\sqrt{n}M_M + \phi_1^2 M_m)^2 - 4\alpha n M_M^2 c \quad (82)$$

Note that $\Delta_3 \geq 0$ would be satisfied as long as $c \leq 0$, that is

$$\alpha\theta_2^2 - 2P_{12}\phi_1\theta_2 + 4P_{12}^2P_{33} - (\phi_1^2 - 4\alpha P_{33})(M_m\alpha - \epsilon_1) \leq 0 \quad (83)$$

Here, we rewrite this inequality with respect to γ as follows

$$\begin{aligned}&((M_m\alpha - \epsilon_1)\alpha^2\bar{\sigma}^2 - \alpha^3\bar{\sigma}^2 - 2P_{12}\alpha^2\bar{\sigma})(\max_i \gamma_i)^2\\&+ c_1\max_i\gamma_i + c_2\min_i\gamma_i + c_3 \geq 0\end{aligned} \quad (84)$$

where c_1, c_2 and c_3 are some certain constants, which are independent with γ_i. It can be founded that this inequality will always hold when γ_i is large enough, under the condition that the coefficient of the second-order term is positive. Generally, the large convergence radius (thus, large ϵ_1 and ϵ_2) is desired, which yields

$$\alpha > \frac{\epsilon_1\bar{\sigma} - 1}{M_m\bar{\sigma} - \bar{\sigma} + \epsilon_1 + \epsilon_2} \quad (85)$$

Therefore from (82), (83), we can get $\sqrt{\Delta_3} > 2\sqrt{n}M_M (\alpha\theta_2 - P_{12}\phi_1) + \phi_1^2 M_m$. Hence, the condition for k_i is as follows

$$\begin{aligned}0 &< \min_i k_i \leq \max_i k_i\\&\leq \frac{2\sqrt{n}M_M(P_{12}\phi_1 - \alpha\theta_2) - \phi_1^2 M_m + \sqrt{\Delta_3}}{2\alpha n M_M^2}\end{aligned} \quad (86)$$

where Δ_3 is defined in (82).

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