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Brief Paper

Stability of an improved dynamic quantised system with time-varying delay and packet losses

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Abstract: The stability problem of a linear dynamic quantised system with time-varying delay and packet losses is studied in this article. An optimal dynamic quantiser which is able to minimise the maximum output error between the quantised system and unquantised systems is designed, and the minimum upper bound of the quantised error is also given. Moreover, the system is lifted into a switched system for stability analysis and a sufficient condition for asymptotic stability is developed in terms of matrix inequalities. Finally, an illustrative example demonstrates the effectiveness of the proposed method.

1 Introduction

In the last few years, studies of quantised control systems have expanded rapidly. An extensive use of computer-aided control systems and the emerging of networked control, where information has to be transmitted through network with limited capacities, becomes important for us to focus studies on data quantisation.

Analysis on quantised control systems was firstly raised in [1], where Kalman firstly studied the quantisation effects. For early studies, efforts were mainly put on interpreting quantised efforts brought by the quantiser. For example, in [2, 3], the quantised variable provides information in a range of values that the unquantised variable may take. As the quantiser is an indispensable part of the quantised control systems, it is necessary to design the quantiser according to the systems. As a result, two representative kinds of quantised control systems have been studied: the static quantised systems and the dynamic quantised systems.

For static quantised control systems, parameters of the quantisers stay invariant when the systems evolve. In [4–6], quadratic stabilisation of discrete-time systems by means of static logarithmic quantisers was studied. Stabilisability of a linear discrete-time system with finite feedback data rates was investigated in [7], where it was shown that the optimal finite horizon coder controller is essentially an optimal quantiser for the initial output, and this work was further developed in [8]. Tradeoffs between quantiser complexity and system performance for scalar systems were analysed in [9, 10]. The coarsest quantiser that can stabilise a discrete-time linear system with stochastic packet loss was derived in [11, 12]. Fridman and Dambrine [13] have studied the design of delayed controller under quantisation where the quantised error was bounded by a given constant. And Wang *et al.* [14] considered H_∞ filtering problem with missing measurements and quantised effects.

Compared with static quantisers, dynamic quantisers generally have better performance as parameters of them vary when the systems evolve. For this reason, dynamic quantised systems have been studied in recent years. In [15] a novel optimal dynamic quantiser was proposed, which was able to minimise the quantised error in the sense of the input–output relation. Besides, the dynamic quantised system considered in [15] performed well and can realise an optimal approximation of a given linear system. Such a quantised system was further studied in [16, 17], where stability of the system is considered. For the dynamic quantiser proposed in [15–17], there exist several examples to demonstrate that it can achieve good control performance [18–21]. Especially, in [21] an experimental comparison has been made between the optimal dynamic quantiser and the static uniform quantiser. Experimental results have confirmed that the optimal dynamic quantiser has a better control performance than the static quantiser. However, asymptotic stability of the quantised system was not investigated in [15–21] because the quantised error cannot be ultimately eliminated. Meanwhile, an effective scaling factor was proposed in [22, 23], using which asymptotic stability of dynamic quantised systems can be achieved. By introducing scaling factor to the optimal dynamic quantiser in [15–17], asymptotic stability can also be guaranteed for the given system. Moreover, time-varying delay and packet loss are not considered in these works, which are usually inevitable especially when signals in the system are transmitted through a communication network [24, 25]. The above observations motivate our study.

The main contributions of this paper are as follows. Firstly, by re-designing parameters of the quantiser proposed in [15–17] and introducing a scaling factor proposed

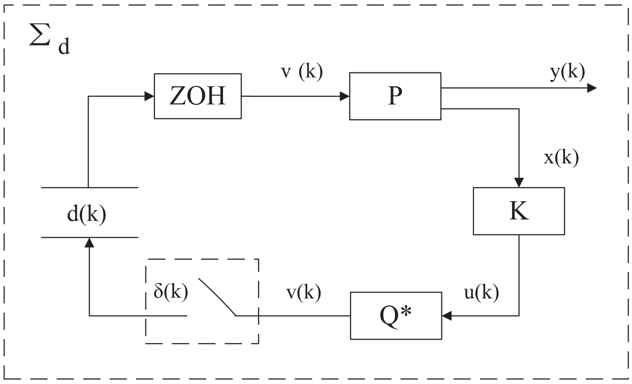

Fig. 1 *Dynamic quantised system with time-varying delay and packet loss*

in [22, 23], a new optimal quantiser that can both minimise the maximum output error and achieve asymptotic stability is obtained. Secondly, networked delay and packet loss have been considered in our paper, which are inevitable for systems with signals transmitted through a communication network. Finally, asymptotic stability has been considered in our paper and a sufficient condition for asymptotic stability is given in Theorem 2.

The paper is organised as follows. Section 2 introduces the dynamic quantised system with time-varying delay and packet losses. Section 3 presents an optimal dynamic quantiser that is able to minimise upper bound of the maximum output error. Section 4 studies stability of our system by 'lifting' it into a switched system [26–31]. Section 5 gives an illustrative example and Section 6 concludes this article.

2 Dynamic quantised system with time-varying delay and packet losses

Consider the discrete-time system as shown in Fig. 1, where the linear plant P is given by

$$P:\begin{cases} x(k+1)=Ax(k)+Bv^*(k)\\ y(k)=Cx(k)\end{cases} \tag{1}$$

where the state $x\in\mathbf{R}^n$, the control input $v^*\in\mathbf{R}^l$ and the output $y\in\mathbf{R}^p$. $A\in\mathbf{R}^{n\times n}$, $B\in\mathbf{R}^{n\times l}$ and $C\in\mathbf{R}^{p\times n}$ are system matrices. The initial state is given as $x(0)=x_0$ for $x_0\in\mathbf{R}^n$.

As depicted in Fig. 1, there exist time-varying delay and packet loss in the feedback channel of the system, where the time-varying delay $d(k)$ satisfies

$$d_1\le d(k)\le d_2 \tag{2}$$

where d_1 and d_2 are known positive integers.

Meanwhile, packet losses here are modelled by the variable $\delta(k)$ which satisfies

$$\delta(k)=\begin{cases} 0 & \text{if } v(k) \text{ is: received}\\ 1 & \text{if } v(k) \text{ is: lost}\end{cases} \tag{3}$$

where $v(k)$ is the output of Q^*.

Assume that the maximum consecutive number of packet losses is m, that is

$$\sum_{l=k-m}^{k}\delta(l)\le m \tag{4}$$

which means that for $m+1$ successive control updates, at least one is available for the actuator.

The dynamic quantiser Q^* used for quantisation can be given as [15]

$$Q^*:\begin{cases} \xi(k+1)=\mathcal{A}\xi(k)+\mathcal{B}(v(k)-u(k))\\ v(k)=q_\mu(\mathcal{C}\xi(k)+u(k))\end{cases} \tag{5}$$

where $\xi\in\mathbf{R}^n$ is the state, $u\in\mathbf{R}^l$ is the input and $v\in\mathbf{V}^l$ is the output, $\mathbf{V}\subset\mathbf{R}$ is the quantisation set.

$\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are system matrices of the quantiser, which will be designed in the next section. The initial state is $\xi(0)=0$, and we set $v(k)=0$ when $u(k)=0$. q_μ in (5) is a scaling quantiser [22] in the form of

$$q_\mu(x)=\begin{cases} \mu(k)M & \text{if } \dfrac{x}{\mu(k)}>M-\dfrac{\Delta}{2}\\ -\mu(k)M & \text{if } \dfrac{x}{\mu(k)}\le -M+\dfrac{\Delta}{2}\\ \left\lfloor \dfrac{x}{\Delta\mu(k)}+\frac{1}{2}\right\rfloor\Delta\mu(k) & \text{if } \dfrac{|x|}{\mu(k)}\le M-\dfrac{\Delta}{2}\end{cases} \tag{6}$$

where Δ is the quantiser sensitivity and M is the saturation value. For $\lfloor a\rfloor=\tilde{a}$, it represents the biggest integer satisfying $\tilde{a}\le a$. $\mu(k)$ is the scaling factor that is monotonically non-increasing, which will be considered later in our paper. For the quantiser (6), it has the following properties

$$\text{If } |x|\le M\mu(k),\ \text{then } |q_\mu(x)-x|\le\frac{\Delta\mu(k)}{2} \tag{7}$$

$$\text{If } |x|> M\mu(k),\ \text{then } |q_\mu(x)|> M\mu(k)-\frac{\Delta\mu(k)}{2} \tag{8}$$

Remark 1: In our paper, by bringing in scaling factor $\mu(k)$ [22, 23] to the static part of the dynamic quantiser raised in [15], the quantised error can be ultimately eliminated, which will be considered later in our paper. Besides, the bringing in of saturation value M makes the quantiser more realisable.

Here $u(k)$ is the output of the state feedback controller which is denoted by

$$u(k)=Kx(k) \tag{9}$$

where $K\in\mathbf{R}^{l\times n}$ is the state feedback gain.

For the input $v^*(k)$ of the plant P, we have the following lemma.

Lemma 1: Consider the dynamic quantised system Σ_d with time-varying delay and packet losses defined in (2)–(4), then the input $v^*(k)$ of P is described by

$$v^*(k)=v\left(\max\{0,k-h_k\}\right) \tag{10}$$

where $h_k\in\{d_1,d_1+1,\ldots,d_2+m\}$ is (see (11))

$$\begin{aligned} h_k=\min\{&d_1+[\delta(k-d_1)+\operatorname{sgn}[\max\{0,d(k-d_1)-d_1\}]](d_2+m),(d_1+1)\\ &+[\delta(k-(d_1+1))+\operatorname{sgn}[\max\{0,d(k-(d_1+1))-(d_1+1)\}]](d_2+m),\ldots,(d_2+m)\\ &+[\delta(k-(d_2+m))+\operatorname{sgn}[\max\{0,d(k-(d_2+m))-(d_2+m)\}]](d_2+m)\}\end{aligned} \tag{11}$$

Proof: The proof is given in the Appendix. □

As is shown in (10), since the output $v(k)$ of the dynamic quantiser (5) is transmitted through the input channel with delay and packet loss, it is necessary for us to choose which signal should be implemented at each instant. Therefore (10) is given to choose the newest control input at each instant for P.

3 Optimal quantiser design

In this section, parameters of the dynamic quantiser (5) are designed to obtain the optimal quantiser that is able to minimise the maximum output error for the system Σ_d.

In Fig. 2, Σ is an usual system with the same P, K as Σ_d and $v(k) = u(k)$. Moreover, both Σ and Σ_d share the same time-varying delay and packet losses process. That is, these two systems are the same at the beginning, and they are equivalent if $v(k) = u(k)$ for Σ_d.

Definition 1: The maximum output error between $y(k)$ and $y^*(k)$ is

$$\mathrm{Er}(Q^*) = \max_{k \in \mathbf{Z}_+} \|y(k, x_0) - y^*(k, x_0)\| \tag{12}$$

where $y(k)$ is the output of Σ_d and $y^*(k)$ is the output of Σ at time k, with the initial state $x(0) = x_0$.

Remark 2: Such a definition of $\mathrm{Er}(Q^*)$ aims at measuring the difference between the output of the dynamic quantised system Σ_d and the system Σ. By designing parameters of (5), we can minimise the upper bound of $\mathrm{Er}(Q^*)$, and realise optimal tracking of the system Σ in the sense of input–output relation.

Throughout this paper, we assume that the following assumption satisfies [15].

Assumption 1: $l = p$ (the dimensions of v and y are the same) and the matrix CB is non-singular.

Remark 3: It is clear that Assumption 1 stands if matrix CB is invertible. There are a number of real-world systems that satisfy this assumption, such as DC servo systems considered in [32, 33], switched bimodal mechanical systems considered in [31] and micro-/nanopositioning system studied in [35]. Besides, for SISO systems this assumption is valid when $CB \neq 0$ [32, 34, 35], which is not difficult to be satisfied.

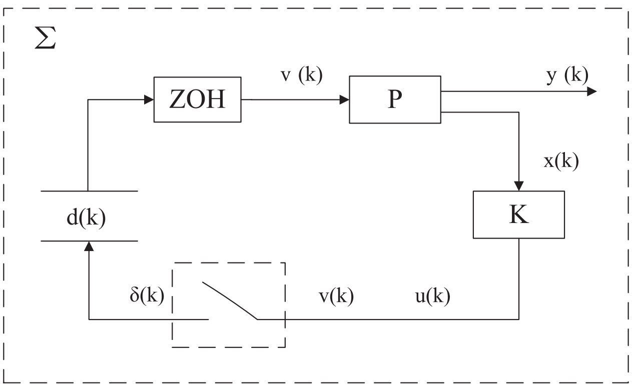

Fig. 2 *Usual system with time-varying delay and packet loss*

For the system Σ_d, an optimal quantiser that is able to minimise the maximum output error can be obtained according to the following theorem.

Theorem 1: The optimal quantiser of the system Σ_d can be given as

$$Q^*: \begin{cases} \xi(k+1) = A\xi(k) - Bu(k) + Bv(k) \\ v(k) = q_\mu(C_Q\xi(k) + u(k)) \end{cases} \tag{13}$$

and the upper bound of the maximum output error can be minimised by

$$\mathrm{Er}(Q^*) \le \|CB\| \frac{\Delta\mu(0)}{2} \tag{14}$$

where $C_Q = -(CB)^{-1}CA$.

Proof: As shown in (10), control input of P at the time instant k is h_k steps later than output of Q^*. Therefore rewrite (5) as

$$Q^*: \begin{cases} \xi(k - h_k + 1) = \mathcal{A}\xi(k - h_k) + \mathcal{B}(v(k - h_k) - u(k - h_k)) \\ v(k - h_k) = q_\mu(\mathcal{C}\xi(k - h_k) + u(k - h_k)) \end{cases} \tag{15}$$

where $h_k \in \{d_1, d_1 + 1, \ldots, d_2 + m\}$.

Let

$$z(k) = \begin{bmatrix} x(k) \\ \xi(k - d_1) \\ \vdots \\ \xi(k - d_2 - m) \end{bmatrix},$$

$$\bar{A}_i = \begin{bmatrix} A & \overbrace{0 & \dots & 0}^{i+d_1-2} & BC & \dots & 0 \\ 0 & \mathcal{A}+\mathcal{BC} & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & \mathcal{A}+\mathcal{BC} & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & \mathcal{A}+\mathcal{BC} \end{bmatrix}$$

$$\bar{A}d = \begin{bmatrix} BK & \dots & 0 \\ 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B & & \\ & \ddots & \\ & & B \end{bmatrix},$$

$$W(i, k) = \begin{bmatrix} \omega(k - i) \\ \omega(k - d_1) \\ \vdots \\ \omega(k - d_2 - m) \end{bmatrix}$$

$$\bar{C} = \begin{bmatrix} C & 0 & \cdots & 0 \end{bmatrix},$$

$$\omega(k) = q_\mu(\mathcal{C}\xi(k) + u(k)) - (\mathcal{C}\xi(k) + u(k))$$

where $i = 1, 2, 3, \ldots, d_2 + m - d_1 + 1$.

Put (1), (9), (10) and (15) together to have Σ_d be given as a switched system

$$\begin{cases} z(k+1) = \bar{A}_{\sigma(k)}z(k) + \bar{A}dz(k - h_k) + \bar{B}W(\sigma(k), k) \\ y(k) = \bar{C}z(k) \end{cases} \tag{16}$$

where the state matrix $\bar{A}_{\sigma(k)}$ switches in the set of possible matrices $\{\bar{A}_{d_1} \ldots \bar{A}_{d_2+m}\}$ according to the parameter

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$\sigma(k)$ called the switching function, which takes value from the finite index set $\mathbf{F} = \{1, 2, 3, \ldots, d_2 + m - d_1 + 1\}$. Moreover, $W(\sigma(k), k)$ is the error vector decided by the pair $(\sigma(k), k)$.

Σ is expressed as

$$\begin{cases} x(k+1) = Ax(k) + BKx(k - h_k) \\ y^*(k) = Cx(k) \end{cases} \tag{17}$$

Therefore difference between $y^*(T, x_0)$ and $y(T, x_0)$ $(T \in \mathbf{Z}_+)$ is

$$\begin{aligned} & y^*(T, x_0) - y(T, x_0) \\ & = CA^T x_0 + C \sum_{j=0}^{s-1} A^j BKx((T - h_k - 1) - j) \\ & \quad - \bar{C}\bar{A}^T_{\sigma(k)} \begin{bmatrix} x_0 & 0 & \ldots & 0 \end{bmatrix}^{\mathrm{T}} \\ & \quad - \bar{C} \sum_{j=0}^{s-1} \bar{A}^j_{\sigma(k)} \bar{A}dz((T - h_k - 1) - j) \\ & \quad - \bar{C} \sum_{l=0}^{T-1} \bar{A}^{(T-1)-l}_{\sigma(k)} \bar{B}W(\sigma(k), l) \\ & = -\bar{C} \sum_{l=0}^{T-1} \bar{A}^{(T-1)-l}_{\sigma(k)} \bar{B}W(\sigma(k), l) \end{aligned} \tag{18}$$

where $s = T - h_k$.

In our paper, by using the 'zooming' method that will be considered in the next section, the scaling factor $\mu(k)$ is designed to guarantee that the quantised saturation will never happen. By designing $\mu(k)$ to be monotonically non-increasing, we can obtain $\mu(0) = \max\{k \in \mathbb{Z} : \mu(k)\}$, which means the quantised error is always smaller than $[\Delta\mu(0)]/2$.

Then it is clear that

$$\begin{aligned} \|y^*(T, x_0) - y(T, x_0)\| & \le \left\| \sum_{l=0}^{T-1} \bar{C}\bar{A}^{(T-1)-l}_{\sigma(k)} \bar{B}\bar{I} \right\| \frac{\Delta\mu(0)}{2} \\ & = \left\| CB + \sum_{l=1}^{T-1} \bar{C}\bar{A}^{(T-1)-l}_{\sigma(k)} \bar{B}\bar{I} \right\| \frac{\Delta\mu(0)}{2} \end{aligned} \tag{19}$$

where $\bar{I} = [I \ \cdots \ I]^{\mathrm{T}}$.

Moreover, when $\mathcal{A} = A$, $\mathcal{B} = B$, $\mathcal{C} = C_Q$, we can obtain $\bar{C}\bar{A}^l_{\sigma(k)}\bar{B}\bar{I} = 0 \ \ (l = 1, 2, \ldots, \infty)$, which means the latter part of (19) is minimised, and therefore it can be given that

$$\mathrm{Er}(Q^*) \le \|CB\| \frac{\Delta\mu(0)}{2} \tag{20}$$

As a result, upper bound of the maximum output error $\mathrm{Er}(Q^*)$ between Σ_d and Σ is minimised, we can obtain the optimal quantiser (13) of the system Σ_d, and the smallest upper bound of $\mathrm{Er}(Q^*)$ is given in (20). □

Remark 4: Theorem 1 is an extension of results in [15], we have proved that quantiser (13) is optimal for our dynamic quantised system Σ_d with time-varying delay and packet losses. The pair $\{\mathcal{A}, \mathcal{B}, \mathcal{C}\}$ in (13) is the same as that in [15]. However, as we use the scaling quantiser q_μ in this paper instead of traditionally static quantiser in [15], the optimal quantiser (13) is different from that in [15].

4 Stability analysis

In this section, a sufficient stability condition for Σ_d is derived by using the lifting method. We employ both Lyapunov function and the 'zooming-in' and 'zooming-out' approaches for stability analysis.

Firstly, let

$$\varsigma(k) = \begin{bmatrix} x(k) \\ \xi(k) \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} A & 0 \\ 0 & A + BC_Q \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}$$

$$\tilde{A}_d = \begin{bmatrix} BK & BC_Q \\ 0 & 0 \end{bmatrix}, \quad \tilde{W}(k) = \begin{bmatrix} \omega(k - h_k) \\ \omega(k) \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} C & 0 \end{bmatrix}$$

system Σ_d can be obtained as

$$\begin{cases} \varsigma(k+1) = \tilde{A}\varsigma(k) + \tilde{A}_d\varsigma(k - h_k) + \tilde{B}\tilde{W}(k) \\ y(k) = \tilde{C}\varsigma(k) \end{cases} \tag{21}$$

Then we have the following definitions

$$\zeta(k) = \begin{bmatrix} \varsigma(k) \\ \varsigma(k-1) \\ \vdots \\ \varsigma(k - d_1) \\ \vdots \\ \varsigma(k - d_2 - m) \end{bmatrix},$$

$$\hat{A}_i = \begin{bmatrix} \tilde{A} & \overbrace{0 & \ldots & 0}^{i+d_1-2} & \tilde{A}d & 0 & \ldots & 0 & 0 \\ I & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & I & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & I & 0 \end{bmatrix},$$

$$\hat{B} = \begin{bmatrix} \tilde{B} & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix}$$

$$\Gamma_i = \begin{bmatrix} 0 & 0 & \overbrace{0 \ldots 0}^{2(i+d_1-2)} & K & C_Q & 0 \ldots 0 \\ K & C_Q & 0 \ldots 0 & 0 & 0 & 0 \ldots 0 \\ 0 & 0 & 0 \ldots 0 & 0 & 0 & 0 \ldots 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 \ldots 0 & 0 & 0 & 0 \ldots 0 \end{bmatrix},$$

$$\hat{C} = \begin{bmatrix} \tilde{C} & 0 & \ldots & 0 \end{bmatrix}, \quad e_\Delta(k) = q_\mu(\Gamma_i\zeta(k)) - \Gamma_i\zeta(k)$$

for $i = 1, 2, 3, \ldots, d_2 + m - d_1 + 1$, where $\hat{A}_i \in \mathbf{R}^{2(d_2+m+1)n \times 2(d_2+m+1)n}$, $\hat{B} \in \mathbf{R}^{2(d_2+m+1)n \times 2l}$ and $\hat{C} \in \mathbf{R}^{p \times 2(d_2+m+1)n}$.

Then system Σ_d can be represented by the following switched system

$$\Sigma_\sigma : \begin{cases} \zeta(k+1) = \hat{A}_{\sigma(k)}\zeta(k) + \hat{B}e_\Delta(k) \\ y(k) = \hat{C}\zeta(k) \\ \zeta(k) = 0 \quad \forall k \le 0 \end{cases} \tag{22}$$

where the state matrix $\hat{A}_{\sigma(k)}$ switches in the set $\{\hat{A}_{d_1} \ldots \hat{A}_{d_2+m}\}$.

Choose a common Lyapunov function for the switching system

$$V(k) = \zeta^{\mathrm{T}}(k)P\zeta(k)$$

where $P \in \mathbf{R}^{2(d_2+m+1)n\times 2(d_2+m+1)n}$ is a positive definite matrix, and $\Delta V(k)$ is

$$\Delta V(k) = V(k+1) - V(k)$$

Before stating the main result, we have the following lemma.

Lemma 2: Given appropriately dimensioned matrices E, F, G, with $E = E^{\mathrm{T}}$. Then

$$E + FG + G^{\mathrm{T}}F^{\mathrm{T}} < 0$$

holds if for some scalar $w > 0$

$$E + w^{-1}FF^{\mathrm{T}} + wG^{\mathrm{T}}G < 0$$

As a result, we can obtain a sufficient stability condition for the switching system (22).

Theorem 2: For a given feedback gain matrix K, packet losses variable $\delta(k)$ and time-varying delay between d_1 and d_2, system (22) is asymptotically stable if there exist $P = P^{\mathrm{T}} > 0$ and $w > 0$ satisfying that

$$(1+w)\hat{A}_i^{\mathrm{T}}P\hat{A}_i - P < 0 \tag{23}$$

where $i = 1, 2, 3, \ldots, d_2 + m - d_1 + 1$.

Proof: Define $\psi = (1 + w^{-1})$, $\hat{I} = [I \quad 0\ldots 0]^{\mathrm{T}}$, $D_i = -[(1+w)\hat{A}_i^{\mathrm{T}}P\hat{A}_i - P]$ and $\lambda_{\min}(D) = \min[\lambda_{\min}(D_i)]$, where $\lambda_{\min}(D_i)$ denotes the smallest eigenvalue of D_i.

Under Lemma 2, it can be obtained that

$$\begin{aligned}\Delta V(k) &= \zeta^{\mathrm{T}}(k+1)P\zeta(k+1) - \zeta^{\mathrm{T}}(k)P\zeta(k)\\ &= \zeta^{T}(k)(\hat{A}_i^{\mathrm{T}}P\hat{A}_i - P)\zeta(k) + 2\zeta^{\mathrm{T}}(k)\hat{A}_i^{\mathrm{T}}P\hat{B}e_{\Delta}(k)\\ &\quad + e_{\Delta}^{\mathrm{T}}(k)\hat{B}^{\mathrm{T}}P\hat{B}e_{\Delta}(k)\\ &\le \zeta^{\mathrm{T}}(k)[(1+w)\hat{A}_i^{\mathrm{T}}P\hat{A}_i - P]\zeta(k)\\ &\quad + (1+w^{-1})e_{\Delta}^{\mathrm{T}}(k)\hat{B}^{\mathrm{T}}P\hat{B}e_{\Delta}(k)\\ &\le -[\lambda_{\min}(D_i)|\zeta(k)|^2 - \psi\|\hat{I}^{\mathrm{T}}\hat{B}^{\mathrm{T}}P\hat{B}\hat{I}\|\Delta^2\mu^2(k)]\\ &\le -[\lambda_{\min}(D)|\zeta(k)|^2 - \psi\|\hat{I}^{\mathrm{T}}\hat{B}^{\mathrm{T}}P\hat{B}\hat{I}\|\Delta^2\mu^2(k)]\end{aligned} \tag{24}$$

where D_i is assumed to satisfy $D_i > 0$ for $i = 1, 2, 3, \ldots, d_2 + m - d_1 + 1$.

The reason for neglecting saturation phenomena in inequality (24) is that saturation is avoided by designing of $\mu(k)$ using the 'zooming' method, which will be considered in the following proof.

It is clear that $\Delta V(k) < 0$ when Theorem 2 is satisfied, and the state of $\sum_d$ will ultimately go inside the region

$$H = \{\zeta(k) : |\zeta(k)| \le \Theta\Delta\mu(k)\} \tag{25}$$

where $\Theta = \sqrt{\{[\psi\|\hat{I}\hat{B}^{\mathrm{T}}P\hat{B}\hat{I}\|]/\{\lambda_{\min}(D)\}\}}$.

Next, we will use the 'zooming' method proposed in [22, 23] to study the convergence from inside H to the equilibrium point of the system.

The zooming-in stage of $\mu(k)$: Let $u(k) = 0$, $\mu(k) = \|A\|^k$. Since the plant P is unstable, it is clear that $\|A\| > 1$. Define

$$k_0 = \min\left\{k \ge 1 : \left\|q_{\mu}\left(\frac{\Gamma_i\zeta(k)}{\mu(k)}\right)\right\| \le M\sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}} - \frac{\Delta}{2}\right\}$$

Initialise $\mu(k)$ to be

$$\mu(k_0) = \|A\|^{k_0} \tag{26}$$

It follows that

$$\left\|\frac{\Gamma_i\zeta(k)}{\mu(k_0)}\right\| \le \left\|q_{\mu}\left(\frac{\Gamma_i\zeta(k)}{\mu(k_0)}\right)\right\| + \frac{\Delta}{2} \le M\sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}$$

Hence

$$\|\Gamma_i\zeta(k)| \le M\mu(k_0)\sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}$$

Let $\|\Gamma\| = \|\Gamma_i\|$, where $i \in \{1, 2, 3, \ldots, d_2 + m - d_1 + 1\}$, we can obtain that

$$|\zeta(k_0)| \le \frac{M}{\|\Gamma\|}\mu(k_0)\sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}$$

Therefore $\zeta(k_0)$ belongs to the region

$$R_1 = \left\{\zeta(k) : \zeta^{\mathrm{T}}(k)P\zeta(k) \le \frac{M^2}{\|\Gamma\|^2}\mu^2(k_0)\lambda_{\min}(P)\right\} \tag{27}$$

It is clear that $\|\Gamma_i\zeta(k)\| \le M\mu(k_0)$ holds for all $\zeta(k) \in R_1$.

Define the scaling factor Ω as

$$\Omega = \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\sqrt{\Theta^2 + \epsilon}\,\|\Gamma\|\Delta M^{-1} \tag{28}$$

where $\epsilon > 0$ is a given parameter. Choose M and Δ in (28) properly to make $\Omega < 1$, then we have $R_1 \supset H$, which means the state of the closed-loop system will never leave R_1.

The zooming-out stage of $\mu(k)$: Define

$$\hat{\tau} = \frac{M^2\lambda_{\min}(P) - \Delta^2\Theta^2\|\Gamma\|^2\lambda_{\max}(P)}{\|\Gamma\|^2\lambda_{\min}(D)\Delta^2\epsilon} \tag{29}$$

it is clear $\hat{\tau} > 0$ as $\Omega < 1$.

Let $\tau = \lfloor\hat{\tau}\rfloor$, assume that

$$\zeta^{\mathrm{T}}(k_0+\tau)P\zeta(k_0+\tau) \le \Delta^2\mu^2(k_0)(\Theta^2+\epsilon)\lambda_{\max}(P) \tag{30}$$

If (30) is not true, we can have

$$\zeta^{\mathrm{T}}(k_0+\tau)P\zeta(k_0+\tau) > \Delta^2\mu^2(k_0)(\Theta^2+\epsilon)\lambda_{\max}(P) \tag{31}$$

Then we have $|\zeta(k_0+\tau)|^2 > \Delta^2\mu^2(k_0)(\Theta^2+\epsilon)$ for all $k \in [k_0, k_0+\tau]$.

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The following inequality can be obtained from (27) and (31)

$$\begin{aligned}&\zeta^{\mathrm{T}}(k_0+\tau)Pz(k_0+\tau)-\zeta^{\mathrm{T}}(k_0)P\zeta(k_0)\\&\quad\geq\Delta^2\mu^2(k_0)(\Theta^2+\epsilon)\lambda_{\max}(P)-\frac{M^2}{\|\Gamma\|^2}\mu^2(k_0)\lambda_{\min}(P)\\&\quad=\lambda_{\max}(P)\Theta^2\Delta^2\mu^2(k_0)-\frac{M^2}{\|\Gamma\|^2}\mu^2(k_0)\lambda_{\min}(P)\end{aligned}\tag{32}$$

Nevertheless, based on (24) and (30) and $\Omega<1$, it is clear that

$$\begin{aligned}&\Delta V(k_0+\tau-1)\\&\quad=\zeta^{\mathrm{T}}(k_0+\tau)P\zeta(k_0+\tau)-\zeta^{\mathrm{T}}(k_0+\tau-1)P\zeta(k_0+\tau-1)\\&\quad\leq-\lambda_{\min}(D)|\zeta(k_0+\tau-1)|^2+\lambda_{\min}(D)\Theta^2\Delta^2\mu^2(k_0)\\&\quad<-\lambda_{\min}(D)\Delta^2\mu^2(k_0)\epsilon\end{aligned}$$

Similarly, it can be obtained that

$$\begin{aligned}&\Delta V(k_0+\tau-j)\\&\quad=\zeta^{\mathrm{T}}(k_0+\tau-j+1)P\zeta(k_0+\tau-j+1)\\&\qquad-\zeta^{\mathrm{T}}(k_0+\tau-j)P\zeta(k_0+\tau-j)\\&\quad\leq-\lambda_{\min}(D)|\zeta(k_0+\tau-j)|^2+\lambda_{\min}(D)\Theta^2\Delta^2\mu^2(k_0)\\&\qquad<-\lambda_{\min}(D)\Delta^2\mu^2(k_0)\epsilon\end{aligned}$$

where $j=\{1,2,3,\ldots,\tau\}$.

Then we have

$$\begin{aligned}&\zeta^{\mathrm{T}}(k_0+\tau)P\zeta(k_0+\tau)-\zeta^{\mathrm{T}}(k_0)P\zeta(k_0)\\&\quad<-\lambda_{\min}(D)\Delta^2\mu^2(k_0)\epsilon\tau\\&\quad\leq-\lambda_{\min}(D)\Delta^2\mu^2(k_0)\epsilon\hat{\tau}\\&\quad=\lambda_{\max}(P)\Theta^2\Delta^2\mu^2(k_0)-\frac{M^2}{\|\Gamma\|^2}\mu^2(k_0)\lambda_{\min}(P)\end{aligned}\tag{33}$$

As (32) and (33) contradict with each other, the validity of (30) has been implied.

Based on (30) and $\Omega<1$, it follows that

$$\begin{aligned}\zeta^{\mathrm{T}}(k_0+\tau)P\zeta(k_0+\tau)&\leq\Delta^2\mu^2(k_0)(\Theta^2+\epsilon)\lambda_{\max}(P)\\&<(\Omega\mu(k_0))^2\frac{M^2}{\|\Gamma\|^2}\lambda_{\min}(P)\end{aligned}$$

Thus, $\zeta(k_0+\tau)$ belongs to

$$R_2=\left\{\zeta(k):\zeta^{\mathrm{T}}(k)P\zeta(k)\leq(\Omega\mu(k_0))^2\frac{M^2}{\|\Gamma\|^2}\lambda_{\min}(P)\right\}$$

Compared with region R_1, it is clear that the radius of region R_2 is smaller, which means the state of the closed-loop system converges after τ steps from k_0.

Let

$$\mu(k)=\Omega^{\lfloor\frac{k-k_0}{\tau}\rfloor}\mu(k_0)\tag{34}$$

where $k\geq k_0$.

For $k_0+\tau\leq k\leq k_0+2\tau$, a similar result can be obtained as

$$\zeta^{\mathrm{T}}(k_0+2\tau)P\zeta(k_0+2\tau)\leq\left(\Omega^2\mu(k_0)\right)^2\frac{M^2}{\|\Gamma\|^2}\lambda_{\min}(P)$$

Moreover, for $k_0+(i-1)\tau\leq k\leq k_0+i\tau$, it can be obtained that

$$\zeta^{\mathrm{T}}(k_0+i\tau)P\zeta(k_0+i\tau)\leq\left(\Omega^i\mu(k_0)\right)^2\frac{M^2}{\|\Gamma\|^2}\lambda_{\min}(P)$$

where the scaling factor $\mu(k)$ can be narrowed every τ steps. That is, $\zeta(k_0+i\tau)$ belongs to

$$R_{i+1}=\left\{\zeta(k):\zeta^{\mathrm{T}}(k)P\zeta(k)\leq\left(\Omega^i\mu(k_0)\right)^2\frac{M^2}{\|\Gamma\|^2}\lambda_{\min}(P)\right\}$$

It is clear that $\mu(k)\to0$ when $k\to\infty$, and $\lim_{k\to\infty}|\zeta(k)|=0$, then the proof of Theorem 2 is completed. □

Remark 5: The zooming-in and zooming-out methods are important to guarantee asymptotical stability of the closed-loop system. As is depicted in (25), when (23) is satisfied, the state outside H will ultimately converge to H. However, as asymptotically stability is considered in our paper, convergence from inside H to the equilibrium point should be further considered. To solve this problem, the zooming-in method is firstly utilised to build region R_1 in (27) that satisfies $R_1\supset H$, which means the state inside H will never leave R_1. Then the zooming-out method is used to decrease radius of R_k $(k=1,2,\ldots)$ gradually. It is clear that the state inside $R_k(k=1,2,\ldots)$ goes to the equilibrium point when radius of the region $R_k(k\to\infty)$ goes to 0, and asymptotic stability of the system is proved.

5 An illustrative example

In this section, an illustrative example is given to illustrate the advantages of the proposed method.

Consider the plant described by

$$A=\begin{bmatrix}1.01&0\\0&0.5\end{bmatrix},\quad B=\begin{bmatrix}-1\\2\end{bmatrix},\quad C=\begin{bmatrix}-1.9&4\end{bmatrix}$$

P is unstable as one of its eigenvalues is outside the unit circle, and it is stabilisable as rank$[B\ AB]=2$.

By using Theorem 1, we can obtain the optimal dynamic quantiser Q^* given as

$$\mathcal{A}=A,\quad\mathcal{B}=-B,\quad\mathcal{C}=\begin{bmatrix}-0.1938&0.202\end{bmatrix}$$

and parameters of quantiser q_μ are given as $M=8\times10^2$ and $\Delta=0.05$.

The system is initialised as $x(0)=[0.3\quad0.5]^{\mathrm{T}}$, $\xi(0)=[0\quad0]^{\mathrm{T}}$, $\epsilon=1\times10^4$, and $\Omega=0.8787<1$.

In our example we let $K=[0.1\quad-0.02]$, $d_1=2$, $d_2=5$ and $m=3$. There exist positive scalar $w=0.1$ and positive definite matrix $P\in\mathbf{R}^{36\times36}$ satisfying (23), then the system Σ_d is asymptotically stable.

The trajectories of the state $x(k)$ under five different methods are given in Figs. 3 and 4, where $x_i(k)$ is the ith component of $x(k)$. Time-varying delay $d(k)$ and packet loss variable $\delta(k)$ are as shown in Figs. 5 and 6, respectively.

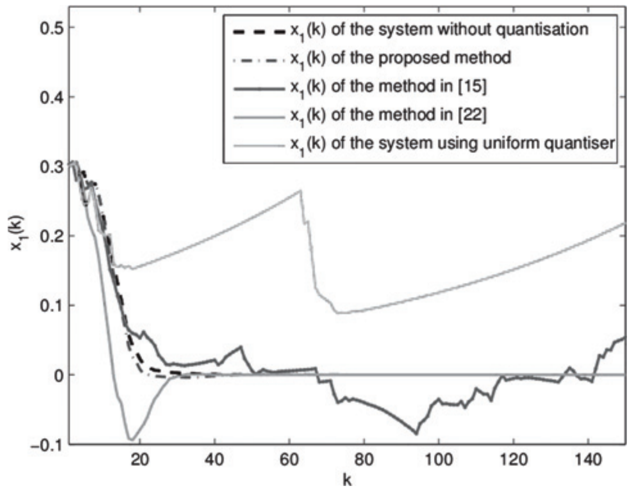

Fig. 3 *Trajectories of the state $x_1(k)$*

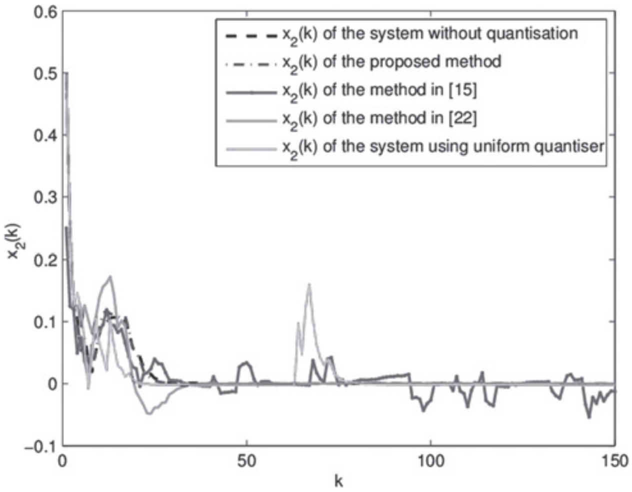

Fig. 4 *Trajectories of the state $x_2(k)$*

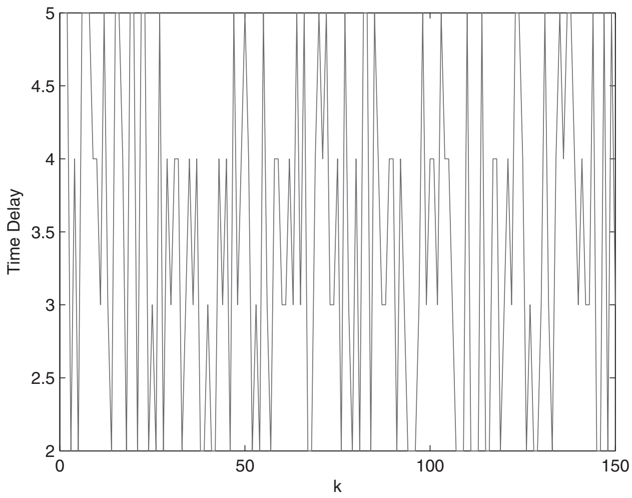

Fig. 5 *Time delay of the system*

As shown in Figs. 3 and 4, $x(k)$ of the proposed method converges to zero, which means system Σ_d is asymptotically stable. However, for method in [15] and that using static uniform quantiser, $x(k)$ does not converge to zero, that is, the system is not asymptotically stable. Although $x(k)$ of

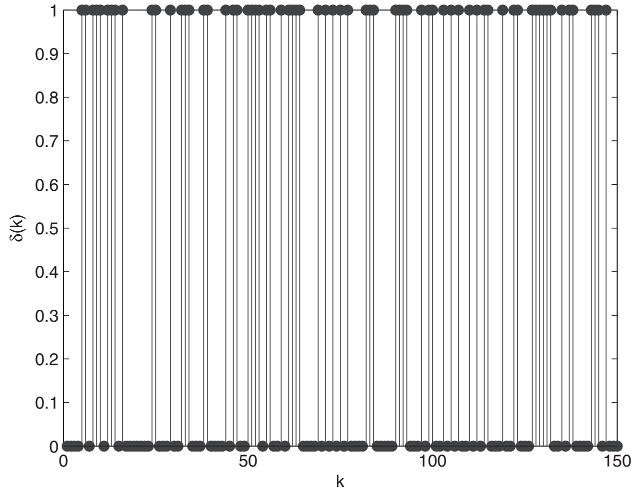

Fig. 6 *Packet loss variable $\delta(k)$*

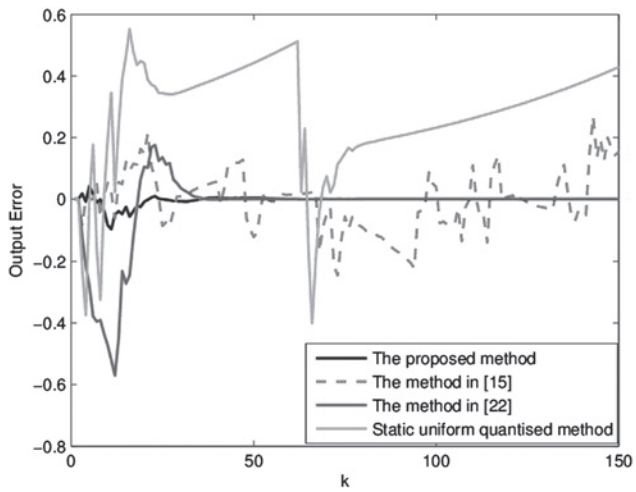

Fig. 7 *Trajectories of output error under four different methods*

the proposed method and of the method in [22] are both converge to zero, the proposed method has a better control performance than those in [22].

A comparison result with regard to output error is given in Fig. 7 to show advantages of the proposed method, where trajectories of the output error $y(k) - y^*(k)$ under four different methods are presented. It is clear that the proposed quantiser has the smallest output error among these four quantisers.

6 Conclusions

The stability problem of a linear dynamic quantised system subject to time-varying delay and packet losses in the feedback channel has been discussed in this paper. Parameters of the original dynamic quantiser are re-designed by combining scaling factor and taking time-varying delays as well as packet losses into consideration to make it optimal for our system. Using the Lyapunov function method, a sufficient condition for asymptotic stability of the system has been obtained in terms of matrix inequalities, and the 'zooming' method is used in the proof to eventually eliminate the steady-state error. The effectiveness of the proposed method has been illustrated through the simulation studies.

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9 Appendix

Proof of Lemma 1: Since the time-varying delay satisfies (2) and the packet loss satisfies (3) and (4), it is clear that the latest input of P which may be available at the time instant k is $v(k-d_1)$, if $v(k-d_1)$ is not lost and suffers the shortest time delay d_1. Meanwhile, the oldest input which may available at the time instant k is $v(k-(d_2+m))$, if it suffers the longest time delay d_2 and the former m successive control updates are lost.

Therefore $v(k-i)$ $(i \in \{d_1, d_1+1, \ldots, d_2+m\})$ are the only inputs of P at the time instant k. Since more than one control update may be available at k, it is necessary for us to make sure which control update should be implemented. Here the newest control input is chosen to be implemented at each instant for P, as is shown in (11).

In this paper we use $\delta(k-i)+\text{sgn}[\max\{0, d(k-i)-i\}]$ to denote whether $v(k-i)$ is available at the time instant k.

If $v(k-i)$ is lost, we can obtain from (3) that $\delta(k-i)=1$ and if the time delay is more than i, then $\text{sgn}[\max\{0, d(k-i)-i\}]=1$. Therefore if and only if $v(k-i)$ is not lost and the delay is less than i that we can obtain $\delta(k-i)+\text{sgn}[\max\{0, d(k-i)-i\}]$ equals to 0, which means it is available at the time instant k. As a result, we can obtain that $v(k-h_k)$ is the newest input of P, where $h_k = \min\{i + [\delta(k-i)+\text{sgn}[\max\{0, d(k-i)-i\}]](d_2+m)\}$ $(i \in \{d_1, d_1+1, \ldots, d_2+m\})$.

Meanwhile, it should be noted that $k-h_k \le 0$ when no control input arrives at the plant P, during which period only $v^*(k)=v(0)$ is available. Therefore we let $v^*(k) = v(\max\{0, k-h_k\})$ to make $v^*(k)=v(0)$ when no control input arrives at P, this completes the proof. □